Minimal dynamical systems on the product of the Cantor set and the circle

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Abstract

We prove that a crossed product algebra arising from a minimal dynamical system on the product of the Cantor set and the circle has real rank zero if and only if that system is rigid. In the case that cocycles take values in the rotation group, it is also shown that rigidity implies tracial rank zero, and in particular, the crossed product algebra is isomorphic to a unital simple AT-algebra of real rank zero. Under the same assumption, we show that two systems are approximately K-conjugate if and only if there exists a sequence of isomorphisms between two associated crossed products which approximately maps $C(X \times \mathbb{T})$ onto $C(X \times \mathbb{T})$.

1 Introduction

A celebrated theorem of Giordano, Putnam and Skau [GPS] gave a dynamical characterization of isomorphism of the crossed product C^*-algebras arising from minimal dynamical systems on the Cantor set. The C^*-algebra theoretic aspect of this result is indebted to the fact that the algebras are unital simple AT-algebras with real rank zero. From the work of Q. Lin and N. C. Phillips [LP2] and the classification of simple C^*-algebras of tracial rank zero (see [L4] and [L5]), crossed product C^*-algebras arising from minimal diffeomorphisms on a manifold are isomorphic if they have the same Elliott invariants in the case that they have real rank zero (see also [LP1]). However, there are no longer any dynamical characterizations of isomorphism of these algebras. The notion of approximate conjugacy was introduced in [LM] where it was also suggested that certain approximate version of conjugacy may be a right equivalence relation to ensure isomorphism of crossed product C^*-algebras. Indeed, a complete description was given for Cantor minimal systems. In the present paper, we consider minimal dynamical systems on the product of the Cantor set X and the circle $\mathbb{T}$, and analyze the associated crossed product C^*-algebras and approximate conjugacy for those systems.

Since the Cantor set X is totally disconnected and the circle $\mathbb{T}$ is connected, every minimal dynamical system on $X \times \mathbb{T}$ can be viewed as a skew product extension of a minimal dynamical system on X. This observation enables us to find a nice "large" subalgebra of the crossed product C^*-algebra in a similar fashion to the Cantor case. By applying results of [Ph2], it will be shown that the algebra has stable rank one and satisfies Blackadar's fundamental comparability property. Moreover, we also prove that the algebra has real rank zero if and only if every invariant measure on X uniquely extends to an invariant measure on $X \times \mathbb{T}$. Such a system is said to be rigid. As a special case we consider cocycles taking their values in the rotation group and show that the algebra has tracial rank zero if and only if the system is rigid. By an easy computation of K-groups and the classification theorem of [L5], we conclude that the associated crossed product C^*-algebras are actually unital simple AT-algebras of real rank zero.

A natural definition of approximate conjugacy for two dynamical systems (X, α) and (Y, β) is the following: there exists a sequence of homeomorphisms $\sigma_n : X \to Y$ such that $\sigma_n \alpha \sigma_n^{-1}$ converges to β in $\mathrm{Homeo}(Y)$. If such a sequence exists, we can construct an asymptotic homomorphism between the associated crossed product C^*-algebras. In [LM], however, it was shown

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that this simple relation is too weak for Cantor minimal systems. This happens because there are no consistency in the sequence $\{\sigma_n\}$. Moreover, we proved in [M3] that a similar result holds for minimal dynamical systems on the product of the Cantor set X and the circle $\mathbb{T}$. To obtain a stronger relation, one should impose additional conditions on the conjugating maps σ_n. We require that σ_n eventually induces the same map on K-theory. Approximate K-conjugacy was introduced in [LM] for Cantor minimal systems, and it was shown that two minimal systems are approximately K-conjugate if and only if the associated crossed products are isomorphic. In the present paper, we will show that two minimal systems on $X \times \mathbb{T}$ associated with cocycles with values in the rotation group are approximately K-conjugate if and only if there is an order and unit preserving isomorphism between the K-theory of two associated crossed products which preserves the images of K-theory of $C(X \times \mathbb{T})$. In fact, in the case that both systems are rigid, when two systems are approximately K-conjugate, the associated crossed products are isomorphic. But more is true. They are C^*-strongly approximately flip conjugate (see Definition 7.4 below). The only difference from the case of Cantor minimal systems is that we have to control a special projection which does not come from any clopen subsets of X. We call that projection the generalized Rieffel projection. Difference of two generalized Rieffel projections will be written by a Bott element associated with two almost commuting unitaries. A technique developed in [M3, Section 4] will be used more carefully in order to fix the position of that projection in the K_0-group.

Non-orientation preserving cases are also discussed. By using a $\mathbb{Z}_2$-extension, we can untwist a non-orientation preserving system and obtain an orientation preserving system. The associated crossed product C^*-algebra turns out to be isomorphic to the fixed point algebra of a $\mathbb{Z}_2$-action on the crossed product associated with the orientation preserving system. Relation of their K-groups is studied.

This paper is organized as follows. In Section 2, we collect notations and terminologies relevant to this paper and establish a few elementary facts. In Section 3, we investigate real rank, stable rank and comparability of projections of the crossed product C^*-algebra. Section 4 is devoted to the case that cocycles take values in the rotation group. In Section 5, when cocycles take values in the rotation group, we will prove that rigidity implies tracial rank zero. In Section 6, the generalized Rieffel projection is defined. In Section 7, we show that isomorphism of K-groups implies approximate K-conjugacy when the systems arise from cocycles with values in the rotation group. In Section 8, we deal with non-orientation preserving cases. In Section 9, we treat examples of various cocycles.

2 Preliminaries

Let A be a unital C^*-algebra. We denote the union of $M_k(A)$ for $k = 1, 2, \ldots$ by $M_\infty(A)$. The K_0-group of A is equipped with the order unit $[1_A]$ and the positive cone $K_0(A)^+$. A homomorphism $s : K_0(A) \to \mathbb{R}$ is called a state if s carries the order unit to one and the positive cone to nonnegative real numbers. We write the set of all states by $S(K_0(A))$ and call it the state space. Let $T(A)$ denote the set of all tracial states on A. Endowed with the weak-$*$ topology, $T(A)$ is a compact convex set. The space of real valued affine continuous functions on $T(A)$ is written by $\mathrm{Aff}(T(A))$. We denote by D the natural homomorphism from $K_0(A)$ to $\mathrm{Aff}(T(A))$.

Namely, $D([p])(\tau)$ is equal to $\tau(p)$, where p is a projection of $M_\infty(A)$ and τ is a tracial state on A. We say that the algebra A satisfies Blackadar's second fundamental comparability question when the order on projections of $M_\infty(A)$ is determined by traces, that is, if $p, q \in M_\infty(A)$ are projections and $\tau(p) < \tau(q)$ for all $\tau \in T(A)$, then p is Murray-von Neumann equivalent to a subprojection of q.

Let X be a compact metrizable space. Equip $\mathrm{Homeo}(X)$ with the topology of pointwise convergence in norm on $C(X)$. Thus a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ in $\mathrm{Homeo}(X)$ converges to α, if

$$\lim_{n\to\infty} \sup_{x\in X} |f(\alpha_n^{-1}(x)) - f(\alpha^{-1}(x))| = 0$$

for every complex valued continuous function $f \in C(X)$. This is equivalent to say that

$$\sup_{x\in X} d(\alpha_n(x), \alpha(x))$$

tends to zero as $n \to \infty$, where $d(\cdot, \cdot)$ is a metric which induces the topology of X. When X is the Cantor set, this is also equivalent to say that, for any clopen subset $U \subset X$, there exists $N \in \mathbb{N}$ such that $\alpha_n(U) = \alpha(U)$ for all $n \geq N$.

When $\alpha : X \to X$ is a homeomorphism on a compact metrizable space X, we denote the crossed product C^*-algebra arising from the dynamical system (X, α) by $C^*(X, \alpha)$. We use the notation $ufu^* = f \circ \alpha^{-1}$, where f is a function on X and u is the implementing unitary. If X is infinite and α is minimal, then $C^*(X, \alpha)$ is a simple C^*-algebra. We regard $C(X)$ as a subalgebra of $C^*(X, \alpha)$. But when we need to emphasize the embedding, it will be denoted by $j_\alpha : C(X) \to C^*(X, \alpha)$. Let M_α denote the set of α-invariant probability measures on X. If α is free, then there exists a canonical bijection between M_α and the tracial state space $T(C^*(X, \alpha))$. We may identify these spaces. Let (Y, β) be another dynamical system. A continuous surjection $F : X \to Y$ is called a factor map if $\beta F = F\alpha$. It is well-known that F yields an affine continuous surjection $F_* : M_\alpha \to M_\beta$ by $F_*(\mu)(E) = \mu(F^{-1}(E))$ for $\mu \in M_\alpha$ and $E \subset Y$. We also remark that F induces a natural embedding of $C^*(Y, \beta)$ into $C^*(X, \alpha)$.

When α is a minimal homeomorphism on the Cantor set X, we call (X, α) a Cantor minimal system. We briefly review results of [GPS]. The crossed product C^*-algebra $C^*(X, \alpha)$ is a unital simple AT algebra of real rank zero, and so it can be classified by its K-groups. The K_1-group of $C^*(X, \alpha)$ is isomorphic to $\mathbb{Z}$, and $K_0(C^*(X, \alpha))$ is unital order isomorphic to

$$K^0(X, \alpha) = C(X, \mathbb{Z})/\{f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z})\}$$

equipped with the positive cone

$$K^0(X, \alpha)^+ = \{[f] : f \in C(X, \mathbb{Z}), f \geq 0\}$$

and the order unit $[1_X]$, where $[f]$ means its equivalence class. We sometimes write $[f]_\alpha$ to specify α. Throughout this paper, $K_0(C^*(X, \alpha))$ will be identified with $K^0(X, \alpha)$. By [GPS, Theorem 2.1], $K^0(X, \alpha)$ is a complete invariant for strong orbit equivalence of Cantor minimal systems. The idea of Kakutani-Rohlin partitions will be used repeatedly. We refer the reader to [HPS, Theorem 4.2] or [M3, Section 2] for details.

We identify the circle with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and write the distance from $t \in \mathbb{T}$ to zero in $\mathbb{T}$ by $|t|$. Let R_t denote the translation on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ by $t \in \mathbb{T}$. Then $\{R_t : t \in \mathbb{T}\}$ forms an abelian subgroup of $\mathrm{Homeo}(\mathbb{T})$. We call it the rotation group. The set of isometric homeomorphisms on $\mathbb{T}$ is written by $\mathrm{Isom}(\mathbb{T})$. Thus,

$$\mathrm{Isom}(\mathbb{T}) = \{R_t : t \in \mathbb{T}\} \cup \{R_t\lambda : t \in \mathbb{T}\},$$

where $\lambda \in \mathrm{Homeo}(\mathbb{T})$ is defined by $\lambda(t) = -t$. The finite cyclic group of order m is denoted by $\mathbb{Z}_m \cong \mathbb{Z}/m\mathbb{Z}$ and may be identified with $\{0, 1, \ldots, m-1\}$.

Define $o : \mathrm{Homeo}(\mathbb{T}) \to \mathbb{Z}_2$ by

$$o(\varphi) = \begin{cases} 0 & \varphi \text{ is orientation preserving} \\ 1 & \varphi \text{ is orientation reversing}. \end{cases}$$

Then the map $o(\cdot)$ is a homomorphism. Let $\mathrm{Homeo}^+(\mathbb{T})$ denote the set of orientation preserving homeomorphisms. Note that $\mathrm{Homeo}^+(\mathbb{T}) \cap \mathrm{Isom}(\mathbb{T})$ consists of rotations.

In the rest of this section, we establish notation and some elementary facts concerning dynamical systems on the product of the Cantor set X and the circle $\mathbb{T}$.

Lemma 2.1. *Let γ be a homeomorphism on $X \times \mathbb{T}$. Then, there exist $\alpha \in \mathrm{Homeo}(X)$ and a continuous map $\varphi : X \to \mathrm{Homeo}(\mathbb{T})$ such that $\gamma(x, t) = (\alpha(x), \varphi_x(t))$ for all $(x, t) \in X \times \mathbb{T}$.*

Proof. This is obvious because the connected component including (x, t) is $\{x\} \times \mathbb{T}$ and it must be carried to a connected component by the homeomorphism γ. $\square$

We denote the homeomorphism of the form in the lemma above by $\alpha \times \varphi$ for short. When $\varphi_x = R_{\xi(x)}$ with a continuous function $\xi : X \to \mathbb{T}$, we write $\alpha \times R_\xi$. Let $F : X \times \mathbb{T} \to X$ be the projection onto the first coordinate. Then we have $F \circ (\alpha \times \varphi) = \alpha \circ F$. Thus, F is a factor map from $(X \times \mathbb{T}, \alpha \times \varphi)$ to (X, α), and so F induces an affine continuous map from the set of invariant measures of $(X \times \mathbb{T}, \alpha \times \varphi)$ to that of (X, α). Note that if $\alpha \times \varphi$ is minimal then α is also minimal.

We say that $\varphi, \psi : X \to \mathrm{Homeo}(\mathbb{T})$ are cohomologous, when there exists a continuous map $\omega : X \to \mathrm{Homeo}(\mathbb{T})$ such that $\psi_x \omega_x = \omega_{\alpha(x)} \varphi_x$ for all $x \in X$. If φ and ψ are cohomologous, it can be easily verified that $\alpha \times \varphi$ and $\alpha \times \psi$ are conjugate.

Let $o(\varphi)$ be the composition of $\varphi : X \to \mathrm{Homeo}(\mathbb{T})$ and $o : \mathrm{Homeo}(\mathbb{T}) \to \mathbb{Z}_2$, that is, $o(\varphi)(x) = o(\varphi_x)$. Under the identification of

$$C(X, \mathbb{Z}_2)/\{f - f\alpha^{-1} : f \in C(X, \mathbb{Z}_2)\}$$

with $K^0(X, \alpha)/2K^0(X, \alpha)$ (see [M1, Lemma 3.5]), an element of $K^0(X, \alpha)/2K^0(X, \alpha)$ is obtained from $o(\varphi)$. We write it by $[o(\varphi)]$ or $[o(\varphi)]_\alpha$.

Definition 2.2. Let (X, α) be a Cantor minimal system and $\varphi : X \to \mathrm{Homeo}(\mathbb{T})$ be a continuous map. We say that $\alpha \times \varphi$ or φ is orientation preserving, if $[o(\varphi)]$ is zero in $K^0(X, \alpha)/2K^0(X, \alpha)$.

Notice that the concept of 'orientation reversing' does not make sense in this situation and that $(\alpha \times \varphi)^2$ may not be orientation preserving.

Lemma 2.3 ([M3, Lemma 4.3]). *Let (X, α) be a Cantor minimal system. If $\alpha \times \varphi$ is an orientation preserving homeomorphism on $X \times \mathbb{T}$, then there exists a continuous map $\psi : X \to \mathrm{Homeo}^+(\mathbb{T})$ such that φ is cohomologous to ψ.*

Let (X, α) be a Cantor minimal system and let $\varphi : X \to \mathrm{Homeo}(\mathbb{T})$ be a continuous map. We would like to compute the K-groups of $C^*(X \times \mathbb{T}, \alpha \times \varphi)$. Let us begin with the orientation preserving case. When $[o(\varphi)]$ is zero in $K^0(X, \alpha)/2K^0(X, \alpha)$, by Lemma 2.3, we may assume that $o(\varphi)(x)$ is zero for all $x \in X$. It is evident that $\alpha \times \varphi$ induces the action

$$\alpha^* : f \mapsto f \circ \alpha^{-1}$$

on $K_0(C(X \times \mathbb{T})) \cong C(X, \mathbb{Z})$, and that the kernel of $\mathrm{id} - \alpha^*$ is spanned by $[1_X]$ and the cokernel of $\mathrm{id} - \alpha^*$ is $K^0(X, \alpha)$. Since $o(\varphi)(x) = 0$ for all $x \in X$, the induced action on $K_1(C(X \times \mathbb{T})) \cong C(X, \mathbb{Z})$ is α^*, too. Consequently we have the following.

Lemma 2.4. *Let (X, α) be a Cantor minimal system and let $\alpha \times \varphi$ be an orientation preserving homeomorphism on $X \times \mathbb{T}$. Then both $K_0(C^*(X \times \mathbb{T}, \alpha \times \varphi))$ and $K_1(C^*(X \times \mathbb{T}, \alpha \times \varphi))$ are isomorphic to $\mathbb{Z} \oplus K^0(X, \alpha)$.*

Of course the embedding $C^*(X, \alpha) \subset C^*(X \times \mathbb{T}, \alpha \times \varphi)$ induces the embedding of $K^0(X, \alpha)$ into $K_0(C^*(X \times \mathbb{T}, \alpha \times \varphi))$ preserving the unit. When $\alpha \times \varphi$ is minimal, it can be easily verified that this is really an order embedding. The order structure of the whole K_0-group will be apparent in later sections.

Next, let us consider the non-orientation preserving case. Suppose that $[o(\varphi)]$ is not zero in $K^0(X, \alpha)/2K^0(X, \alpha)$. Clearly $\alpha \times \varphi$ induces the same action on $K_0(C(X \times \mathbb{T})) \cong C(X, \mathbb{Z})$ as the orientation preserving case. But, on the K_1-group, the induced action is different. It is given by

$$\alpha_\varphi^*(f)(x) = (-1)^{o(\varphi)(\alpha^{-1}(x))} f(\alpha^{-1}(x))$$

for $f \in C(X, \mathbb{Z})$. We need to know the kernel and the cokernel of $\mathrm{id} - \alpha_\varphi^*$. Suppose that $f \in C(X, \mathbb{Z})$ belongs to $\mathrm{Ker}(\mathrm{id} - \alpha_\varphi^*)$ and $f \neq 0$. Since $|f(x)| = |\alpha_\varphi^*(f)(x)| = |\alpha^*(f)(x)|$ for all $x \in X$, the minimality of α implies that $|f(x)|$ is a constant function. Define $c \in C(X, \mathbb{Z}_2)$ by $f(x) = (-1)^{c(x)}|f(x)|$. Then $f(x) = (-1)^{o(\varphi)(\alpha^{-1}(x))} f(\alpha^{-1}(x))$ yields $c(x) = o(\varphi)(\alpha^{-1}(x)) + c(\alpha^{-1}(x))$, which contradicts $[o(\varphi)] \neq 0$. Hence $\mathrm{id} - \alpha_\varphi^*$ is injective. It follows that $K_0(C^*(X \times \mathbb{T}, \alpha \times \varphi)) \cong K^0(X, \alpha)$.

Let $(X \times \mathbb{Z}_2, \alpha \times o(\varphi))$ be the skew product extension associated with the $\mathbb{Z}_2$-valued cocycle $o(\varphi)$. As $[o(\varphi)] \neq 0$, $(X \times \mathbb{Z}_2, \alpha \times o(\varphi))$ is a Cantor minimal system (see [M1, Lemma 3.6]). By definition, $K^0(X \times \mathbb{Z}_2, \alpha \times o(\varphi))$ is isomorphic to the cokernel of $\mathrm{id} - (\alpha \times o(\varphi))^*$ on $C(X \times \mathbb{Z}_2, \mathbb{Z})$. Let π be the projection from $X \times \mathbb{Z}_2$ onto the first coordinate. Then π is a factor map from $(X \times \mathbb{Z}_2, \alpha \times o(\varphi))$ to (X, α). It is well-known that $[f] \mapsto [f \circ \pi]$ is an order embedding between the K^0-groups, and so we will regard $K^0(X, \alpha)$ as a subgroup of $K^0(X \times \mathbb{Z}_2, \alpha \times o(\varphi))$. It is convenient to introduce a monomorphism δ from $C(X, \mathbb{Z})$ to $C(X \times \mathbb{Z}_2, \mathbb{Z})$ defined by $\delta(f)(x, k) = (-1)^k f(x)$. Then we can check that

$$\delta \circ \alpha_\varphi^* = (\alpha \times o(\varphi))^* \circ \delta.$$

Define $\gamma \in \mathrm{Homeo}(X \times \mathbb{Z}_2)$ by $\gamma(x, k) = (x, k+1)$. Note that γ commutes with $\alpha \times o(\varphi)$. Since $\mathrm{Im}\, \delta = \mathrm{Im}(\mathrm{id} - \gamma^*)$, we have

$$
\begin{aligned}
\mathrm{Coker}(\mathrm{id} - \alpha_\varphi^*) &\cong \mathrm{Im}\, \delta / \mathrm{Im}\, \delta \circ (\mathrm{id} - \alpha_\varphi^*) \\
&= \mathrm{Im}(\mathrm{id} - \gamma^*) / \mathrm{Im}((\mathrm{id} - (\alpha \times o(\varphi))^*) \circ (\mathrm{id} - \gamma^*)) \\
&\cong C(X \times \mathbb{Z}_2, \mathbb{Z}) / (\mathrm{Ker}(\mathrm{id} - \gamma^*) + \mathrm{Im}(\mathrm{id} - (\alpha \times o(\varphi))^*)) \\
&\cong K^0(X \times \mathbb{Z}_2, \alpha \times o(\varphi)) / K^0(X, \alpha).
\end{aligned}
$$

We can summarize the conclusion just obtained as follows.

Lemma 2.5. *Let (X, α) be a Cantor minimal system and let $\alpha \times \varphi$ be a non-orientation preserving homeomorphism on $X \times \mathbb{T}$. Then we have*

$$K_0(C^*(X \times \mathbb{T}, \alpha \times \varphi)) \cong K^0(X, \alpha)$$

and

$$K_1(C^*(X \times \mathbb{T}, \alpha \times \varphi)) \cong \mathbb{Z} \oplus K^0(X \times \mathbb{Z}_2, \alpha \times o(\varphi)) / K^0(X, \alpha),$$

where the $\mathbb{Z}$-summand of the K_1-group is generated by the implementing unitary.

By the same reason as orientation preserving cases, one sees that $K_0(C^*(X \times \mathbb{T}, \alpha \times \varphi))$ is unital order isomorphic to $K^0(X, \alpha)$, when $\alpha \times \varphi$ is minimal.

We remark that the torsion subgroup of

$$K^0(X \times \mathbb{Z}_2, \alpha \times o(\varphi))/K^0(X, \alpha)$$

is isomorphic to $\mathbb{Z}_2$ by [M3, Lemma 4.5]. The torsion element is given by

$$f_0(x, k) = \begin{cases} 1 & o(\varphi)(\alpha^{-1}(x)) = 1 \text{ and } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

3 Real rank and stable rank

Let (X, α) be a Cantor minimal system and let $X \ni x \mapsto \varphi_x \in \mathrm{Homeo}(\mathbb{T})$ be a continuous map. In this section, we would like to compute the real rank and the stable rank of $C^*(X \times \mathbb{T}, \alpha \times \varphi)$. Let us denote the crossed product C^*-algebra by A and the implementing unitary by u. Our crucial tool is a "large" subalgebra of A, which is described below. The proof will be done by applying the argument of [Ph2] to that subalgebra.

We would like to begin with the definition of the rigidity.

Definition 3.1. Let $\alpha \times \varphi$ be a homeomorphism on the product of the Cantor set X and the circle $\mathbb{T}$. We say that $\alpha \times \varphi$ is rigid, if the canonical factor map from $(X \times \mathbb{T}, \alpha \times \varphi)$ to (X, α) induces an isomorphism between the sets of invariant probability measures.

Remark 3.2. In the definition above, even if α is minimal and $\alpha \times \varphi$ is rigid, $\alpha \times \varphi$ may not be minimal. For example, let (X, α) be an odometer system and let $\varphi \in \mathrm{Homeo}^+(\mathbb{T})$ be a Denjoy homeomorphism, that is, the rotation number of φ is irrational but φ is not conjugate to a rotation. It is well-known that φ has a unique invariant nontrivial closed subset Y, and so $\alpha \times \varphi$ has a nontrivial closed invariant subset $X \times Y$. Thus $\alpha \times \varphi$ is not minimal. It is also known that Y is the Cantor set and $\varphi|Y$ is minimal. This Cantor minimal system is called a Denjoy system (see [PSS] for details). The complement of Y consists of countable disjoint open intervals and each interval is a wandering set. Hence, for every $\alpha \times \varphi$-invariant probability measure μ, we have $\mu(X \times Y^c) = 0$. Moreover, as pointed out in [M1, Section 7 (2)], the product of (X, α) and $(Y, \varphi|Y)$ is uniquely ergodic. It follows that $\alpha \times \varphi$ is uniquely ergodic. In particular, it is rigid.

For $x \in X$, let A_x be the C^*-subalgebra generated by $C(X \times \mathbb{T})$ and $uC_0((X \setminus \{x\}) \times \mathbb{T})$. In [Pu1, Theorem 3.3], it was proved that $A_x \cap C^*(X, \alpha)$ is a unital AF algebra, where we regard $C^*(X, \alpha)$ as a C^*-subalgebra of A. This AF subalgebra played a crucial role in the subsequent papers [HPS] and [GPS]. In our situation, the C^*-subalgebra A_x is not an AF algebra but an AT algebra, and it helps us to show real rank zero.

Proposition 3.3. *In the setting above, we have the following.*

(1) *A_x is a unital AT algebra.*

(2) *When $\alpha \times \varphi$ is orientation preserving, $K_0(A_x)$ is unital order isomorphic to $K^0(X, \alpha)$ and $K_1(A_x)$ is isomorphic to $K^0(X, \alpha)$.*

(3) *When $\alpha \times \varphi$ is not orientation preserving, $K_0(A_x)$ is unital order isomorphic to $K^0(X, \alpha)$ and $K_1(A_x)$ is isomorphic to an extension of $\mathrm{Coker}(\mathrm{id} - \alpha_\varphi^*)$ by $\mathbb{Z}$.*

(4) *There exists a canonical bijection between the tracial state space $T(A_x)$ and the set of $\alpha \times \varphi$-invariant probability measures.*

(5) *A_x is simple if and only if $\alpha \times \varphi$ is minimal.*

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(6) *If $\alpha \times \varphi$ is minimal, then A_x is real rank zero if and only if $\alpha \times \varphi$ is rigid.*

Proof. (1) Let
$$\mathcal{P}_n = \{X(n,v,k) : v \in V_n, k = 1, 2, \ldots, h(v)\}$$
be a sequence of Kakutani-Rohlin partitions which gives a Bratteli-Vershik model for (X, α) (see [HPS, Theorem 4.2] or [M3, Section 2] for Kakutani-Rohlin partitions). We assume that the sequence of the roof sets
$$R(\mathcal{P}_n) = \bigcup_{v \in V} X(n, v, h(v))$$
shrinks to a singleton $\{x\}$. Let A_n be the C^*-subalgebra generated by $C(X \times \mathbb{T})$ and $uC(R(\mathcal{P}_n)^c \times \mathbb{T})$. It is easy to see that A_x is the norm closure of the union of all A_n's. By using a similar argument to [Pu1, Lemma 3.1], it can be shown that A_n is isomorphic to
$$\bigoplus_{v \in V_n} M_{h(v)} \otimes C(X(n, v, h(v))) \otimes C(\mathbb{T}),$$
which is an AT algebra. To verify this, define a projection p_v by
$$p_v = \sum_{k=1}^{h(v)} 1_{X(n,v,k)}$$
for each $v \in V$. Since $p_v u(1 - 1_{R(\mathcal{P}_n)}) = u(1 - 1_{R(\mathcal{P}_n)})p_v$, the projection p_v is central in A_n. Clearly $u^{i-j}1_{X(n,v,j)}$ for $i, j = 1, 2, \ldots, h(v)$ form matrix units of $p_v A_n p_v$. By
$$1_{X(n,v,h(v))} A_n 1_{X(n,v,h(v))} = C(X(n, v, h(v)) \times \mathbb{T}),$$
we obtain the description above. Hence A_x is also an AT algebra.

(2) There is a natural homomorphism from $K_i(C(X \times \mathbb{T})) \cong C(X, \mathbb{Z})$ to $K_i(A_n)$ for $i = 1, 2$. By (1), the kernel of this map is
$$\{f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}), f(y) = 0 \text{ for all } y \in R(\mathcal{P}_n)\}.$$
Therefore $K_i(A_x)$ is isomorphic to
$$C(X, \mathbb{Z})/\{f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}), f(x) = 0\}.$$
It follows from $1_X \circ \alpha^{-1} = 1_X$ that
$$\{f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}), f(x) = 0\} = \{f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z})\},$$
which implies $K_i(A_x) \cong K^0(X, \alpha)$.

(3) The computation of K_0-group is the same as the orientation preserving case. Let us consider $K_1(A_x)$. It is not hard to see that $K_1(A_x)$ is isomorphic to
$$C(X, \mathbb{Z})/\{f - \alpha_\varphi^*(f) : f \in C(X, \mathbb{Z}) \text{ and } f(x) = 0\}.$$
We follow the notation used in the discussion before Lemma 2.5. The image of
$$\{f - \alpha_\varphi^*(f) : f \in C(X, \mathbb{Z}) \text{ and } f(x) = 0\}$$
by δ is equal to the image of
$$\{f - f \circ (\alpha \times o(\varphi))^{-1} : f \in C(X \times \mathbb{Z}_2, \mathbb{Z}), f(x, 0) = (x, 1) = 0\}$$

by id $-\gamma^*$. Hence, in the same way as Lemma 2.5, we have

$$K_1(A_x) \cong K^0(X \times \mathbb{Z}_2, \alpha \times o(\varphi); \mathbb{Z}_2)/K^0(X, \alpha).$$

See [M2] for the definition of $K^0(X \times \mathbb{Z}_2, \alpha \times o(\varphi); \mathbb{Z}_2)$. By [Pu1, Theorem 4.1],

$$0 \to \mathbb{Z} \to K^0(X \times \mathbb{Z}_2, \alpha \times o(\varphi); \mathbb{Z}_2) \to K^0(X \times \mathbb{Z}_2, \alpha \times o(\varphi)) \to 0$$

is exact, which implies that $K_1(A_x)$ is an extension of

$$\mathrm{Coker}(\mathrm{id} - \alpha_\varphi^*) \cong K^0(X \times \mathbb{Z}_2, \alpha \times o(\varphi))/K^0(X, \alpha)$$

by the integers $\mathbb{Z}$.

(4) Since there exists a canonical bijection between $T(A)$ and the set of $\alpha \times \varphi$-invariant probability measures, it suffices to check that every $\tau \in T(A_x)$ extends to a tracial state on A. Let $f \in C(X \times \mathbb{T})$ be a function satisfying $0 \leq f \leq 1$. Take a natural number $n \in \mathbb{N}$ arbitrarily. We can find a clopen neighborhood U of x such that $U, \alpha^{-1}(U), \ldots, \alpha^{-n}(U)$ are mutually disjoint. Let $p = 1_{U \times \mathbb{T}} \in C(X \times \mathbb{T})$. Then we have $\tau(p) < n^{-1}$, because $p, u^*pu, \ldots, u^{n*}pu^n$ are mutually equivalent in A_x and mutually disjoint. We also notice that $u(1-p)f$ belongs to A_x. It follows that

$$\tau(f) \leq \tau(p) + \tau((1-p)f) = \tau(p) + \tau(u(1-p)fu^*) < \frac{1}{n} + \tau(ufu^*).$$

Similarly we can see $\tau(ufu^*) < n^{-1} + \tau(f)$. Since n is arbitrary, $\tau(f)$ equals $\tau(ufu^*)$, which means that τ extends to a trace on A.

(5) Note that the C^*-algebra A can be regarded as a groupoid C^*-algebra associated with the equivalence relation

$$\mathcal{R} = \{(z, (\alpha \times \varphi)^k(z)) : z \in X \times \mathbb{T}, k \in \mathbb{Z}\}.$$

Then the C^*-subalgebra A_x corresponds to the subequivalence relation

$$\mathcal{R}_x = \mathcal{R} \setminus \{((\alpha \times \varphi)^k(x, t), (\alpha \times \varphi)^l(x, t)) : t \in \mathbb{T}, (1-k, l) \in \mathbb{N}^2 \text{ or } (k, 1-l) \in \mathbb{N}^2\}.$$

It is well-known that a homeomorphism on a compact space is minimal if and only if every positive orbit is dense. Therefore $\alpha \times \varphi$ is minimal if and only if each equivalence class of $\mathcal{R}_x$ is dense in $X \times \mathbb{T}$. It follows from [R, Proposition 4.6] that this is equivalent to A_x being simple.

(6) By (1) and (5), A_x is a unital simple AT algebra. By (2) and (3), in the K_0-group, every projection of A_x is equivalent to some $[f] \in K^0(X, \alpha)$. Hence, projections in A_x separate traces on A_x if and only if $\alpha \times \varphi$ is rigid. Then the conclusion follows from [BBEK, Theorem 1.3]. $\square$

Remark 3.4. The six-term exact sequence of [Pu2, Theorem 2.4] applies to this situation. See [Pu2, Example 2.6]. The reduced groupoid C^*-algebra $C_r^*(H)$ appearing there is isomorphic to $C(\mathbb{T}) \otimes \mathcal{K}$.

We would like to consider the real rank of A. In [Ph2], it was shown that if G is an almost AF Cantor groupoid and $C_r^*(G)$ is simple, then $C_r^*(G)$ has real rank zero. The key of its proof was the presence of a "large" AF subalgebra. We will show that a similar argument is possible when the AF subalgebra is replaced by a subalgebra with tracial rank zero.

Definition 3.5 ([L5, Theorem 6.13]). We say that a unital simple C^*-algebra A has tracial (topological) rank zero, if for any finite subset $\mathcal{F} \subset A$, any $\varepsilon > 0$ and any nonzero positive element $c \in A$, there exists a projection $e \in A$ and a finite dimensional unital subalgebra $E \subset eAe$ (that is, e is the identity of E) such that:

(1) $\|ae - ea\| < \varepsilon$ for all $a \in \mathcal{F}$.

(2) For every $a \in \mathcal{F}$, there is $b \in E$ such that $\|pap - b\| < \varepsilon$.

(3) $1 - e$ is Murray-von Neumann equivalent to a projection in $\overline{cAc}$.

Definition 3.6. A unital simple C^*-algebra A is called an almost tracially AF algebra, if there exists a unital simple subalgebra $B \subset A$ with tracial rank zero such that the following holds: for any finite subset $\mathcal{F} \subset A$ and any $\varepsilon > 0$, there exists a projection $p \in B$ such that:

(1) For every $a \in \mathcal{F}$, there is $b \in B$ such that $\|ap - b\| < \varepsilon$.

(2) $\tau(1 - p) < \varepsilon$ for every tracial state $\tau \in T(B)$.

Lemma 3.7. *Suppose that $\alpha \times \varphi$ is a minimal homeomorphism on $X \times \mathbb{T}$. If $\alpha \times \varphi$ is rigid, then the crossed product C^*-algebra $A = C^*(X \times \mathbb{T}, \alpha \times \varphi)$ is an almost tracially AF algebra.*

Proof. Take $x \in X$. By Proposition 3.3, A_x is a unital simple AT algebra of real rank zero. It follows from [L1, Proposition 2.6] that A_x is a unital simple algebra with tracial rank zero. Suppose that a finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$ are given. Since

$$\left\{ \sum_{k=-N}^{N} f_k u^k : N \in \mathbb{N}, f_k \in C(X \times \mathbb{T}) \right\}$$

is a dense subalgebra of A, we may assume that there is $N \in \mathbb{N}$ such that $\mathcal{F}$ is contained in

$$E_N = \left\{ \sum_{k=-N}^{N} f_k u^k : f_k \in C(X \times \mathbb{T}) \right\}.$$

We can find a clopen neighborhood U of x so that $\alpha^{1-N}(U), \alpha^{2-N}(U), \ldots, U, \alpha(U), \ldots, \alpha^N(U)$ are mutually disjoint and $\mu(U) < \varepsilon/2N$ for all $\mu \in M_\alpha$. Put $V = \bigcup_{k=1-N}^{N} \alpha^k(U)$ and $p = 1_{V^c \times \mathbb{T}} \in C(X \times \mathbb{T})$. It is easy to see that $u^k p$ and $u^{1-k} p$ belong to A_x for $k = 0, 1, \ldots, N$, and so $ap \in A_x$ for every $a \in E_N$. Moreover, $\tau(1 - p)$ is less than ε for every $\tau \in T(A_x)$. This finishes the proof of the lemma. $\square$

The following lemma is a generalization of [Ph2, Lemma 4.3].

Lemma 3.8. *Let A be a unital simple algebra with tracial rank zero. Let $p \in A$ be a projection and let $a \in A$ be a nonzero self-adjoint element. For any $\varepsilon > 0$ and $n \in \mathbb{N}$ satisfying $n\varepsilon > 1$, there exists a projection $q \in A$ such that $\|qa - aq\| < \varepsilon\|a\|$, $p \leq q$ and*

$$\tau(q) < (2n + 1)\tau(p)$$

for all $\tau \in T(A)$.

Proof. Without loss of generality, we may assume $\|a\| = 1$. Put $\varepsilon_0 = 8^{-1}(\varepsilon - n^{-1})$. Choose $\delta_0 > 0$ so that whenever $p, q \in A$ are projections satisfying $\|pq - p\| < \delta_0$, then there exists a unitary $u \in A$ such that

$$\|u - 1\| < \varepsilon_0 \quad \text{and} \quad p \leq uqu^*.$$

Let

$$\delta = \min\{\varepsilon_0, 4^{-1}\delta_0, (2n + 1)^{-1}\tau(p) : \tau \in T(A)\}.$$

By [L1, Proposition 2.4] together with results of [L1, Section 3], there exist a projection $e \in A$ and a finite dimensional unital subalgebra $E \subset eAe$ such that:

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- $\|ae - ea\| < \delta$ and $\|pe - ep\| < \delta$.

- There exist $b, c \in E$ such that $\|eae - b\| < \delta$ and $\|epe - c\| < \delta$.

- $\tau(1 - e) < \delta$ for every tracial state $\tau \in T(A)$.

We may assume that $b = b^*$, $\|b\| = 1$ and c is a projection. Thanks to [Ph2, Lemma 4.2], there exists a projection $q_0 \in E$ such that

$$c \le q_0, \ \ [q_0] \le 2n[c] \in K_0(E), \ \ \text{and} \ \ \|q_0 b - b q_0\| < \frac{1}{n}.$$

Put $q = 1 - e + q_0$. Since

$$\begin{aligned}
&\|pq - p\| \\
&\le \|(p(1 - e) + peq_0) - (p(1 - e) + pcq_0)\| + \|(p(1 - e) + pcq_0) - (p(1 - e) + pe)\| \\
&< 2\delta + 2\delta \le \delta_0,
\end{aligned}$$

there is a unitary $u \in A$ such that $\|u - 1\| < \varepsilon_0$ and $p \le uqu^*$. It is not hard to see that

$$\begin{aligned}
\tau(uqu^*) = \tau(q) &= \tau(1 - e) + \tau(q_0) \\
&< \delta + 2n\tau(c) < (2n + 1)\delta + 2n\tau(pe) < (2n + 1)\tau(p)
\end{aligned}$$

for all $\tau \in T(A)$ and that

$$\begin{aligned}
\|[uqu^*, a]\| &< \|[q, a]\| + 4\varepsilon_0 \\
&< \|[q, b]\| + 4\delta + 4\varepsilon_0 \\
&< \frac{1}{n} + 8\varepsilon_0 = \varepsilon,
\end{aligned}$$

thereby completing the proof. $\qquad\square$

The following is a well-known matrix trick. We omit the proof.

Lemma 3.9. *When a is a self-adjoint element of a unital C^*-algebra A,*

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$$

is approximated by an invertible self-adjoint element in $A \otimes M_2$.

Although the proof of the following theorem is almost the same as that of [Ph2, Theorem 4.7], we would like to state it for the reader's convenience.

Theorem 3.10. *If a unital simple C^*-algebra A is an almost tracially AF algebra, then A has real rank zero.*

Proof. Let $B \subset A$ be a unital simple subalgebra with tracial rank zero as in Definition 3.6.

Let $a \in A$ be self-adjoint and non-invertible. It suffices to show that a is approximated by a self-adjoint invertible element of A. Without loss of generality, we may assume $\|a\| \le 1$. Take $\varepsilon > 0$ arbitrarily. Define a continuous function g on $[-1, 1]$ by

$$g(t) = \begin{cases} 1 - \varepsilon^{-1}|t| & |t| \le \varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

Put
$$\varepsilon_0 = \min\{\tau(g(a)) : \tau \in T(A)\}.$$

Since A is simple, ε_0 is positive. Applying [Ph2, Lemma 4.4] to $g : [-1, 1] \to [0, 1]$ and $4^{-1}\varepsilon_0 > 0$, we obtain $\delta > 0$. We may assume that δ is less than ε. Choose a natural number $n \in \mathbb{N}$ so that

$$\frac{1}{n} < \min\left\{\frac{\varepsilon_0}{12}, \frac{\delta}{2}\right\}.$$

By definition, there is a projection $p \in B$ such that $a(1 - p)$ is close to B and $\tau(p)$ is less than n^{-2} for all $\tau \in T(B)$. By perturbing a, we may assume that $a(1 - p)$ belongs to B. Then, we can apply Lemma 3.8 to $a - pap \in B$ and $p \in B$, and get a projection $q \in B$ such that $p \leq q$,

$$\|[q, a - pap]\| < \frac{\delta}{2}\|a - pap\| \leq \delta$$

and

$$\tau(q) \leq (2n + 1)\tau(p) < \frac{3}{n} < \frac{\varepsilon_0}{4}$$

for every tracial state $\tau \in T(B)$. It follows that

$$\tau(g(a)) - \tau(q) > \varepsilon_0 - \frac{\varepsilon_0}{4} > \frac{\varepsilon_0}{4}.$$

We also notice that $\|[q, a]\| < \delta$. Put $a_0 = (1 - q)a(1 - q) \in B$. By the choice of δ, we have

$$\tau(g(a_0)) > \tau(g(a)) - \tau(q) - \frac{\varepsilon_0}{4} > \frac{\varepsilon_0}{2}.$$

A unital simple algebra with tracial rank zero has real rank zero by [L1, Theorem 3.4], and so there exists a projection r in the hereditary subalgebra of B generated by $g(a_0)$ such that

$$\|rg(a_0) - g(a_0)\| < \frac{\varepsilon_0}{4}.$$

Then

$$\tau(r) \geq \tau(rg(a_0)r) > \tau(g(a_0)) - \frac{\varepsilon_0}{4} > \frac{\varepsilon_0}{4}$$

for all $\tau \in T(B)$. Since the order on projections of B is determined by traces (see [L2, Theorem 6.8, 6.13]), there is a projection $r_0 \in B$ such that $r_0 \leq r$ and $r_0 \sim q$. Moreover, by means of [Ph2, Lemma 4.6], we have

$$\|r_0 a_0 - a_0 r_0\| < 2\varepsilon \quad \text{and} \quad \|r_0 a_0 r_0\| < \varepsilon.$$

As a result,

$$a \overset{2\varepsilon}{\approx} qaq + a_0$$
$$\overset{4\varepsilon}{\approx} qaq + r_0 a_0 r_0 + (1 - q - r_0)a_0(1 - q - r_0)$$
$$\overset{\varepsilon}{\approx} qaq + (1 - q - r_0)a_0(1 - q - r_0)$$

is obtained. The element $(1 - q - r_0)a_0(1 - q - r_0)$ belongs to B and B has real rank zero. By applying Lemma 3.9 to qaq and $r_0 \sim q$, we can get the conclusion. $\qquad\square$

Corollary 3.11. *For a minimal homeomorphism $\alpha \times \varphi$ on $X \times \mathbb{T}$, the following are equivalent.*

(1) $\alpha \times \varphi$ *is rigid.*

(2) *The crossed product C^*-algebra $A = C^*(X \times \mathbb{T}, \alpha \times \varphi)$ has real rank zero.*

(3) *$D(K_0(A))$ is uniformly dense in $\mathrm{Aff}(T(A))$.*

Proof. (1)$\Rightarrow$(2). This is immediate from Lemma 3.7 and the theorem above.

(2)$\Rightarrow$(3). Since $\alpha \times \varphi$ is minimal, A is simple. By Theorem 3.12, A is stably finite and the projections in $A \otimes \mathcal{K}$ satisfy cancellation. Furthermore, $K_0(A)$ is weakly unperforated by [Ph1, Theorem 4.5]. It follows from [B, Theorem 6.9.3] that the image of $K_0(A)$ is uniformly dense in real valued affine continuous functions on $QT(A)$, the set of quasitraces on A. Because every element of $\mathrm{Aff}(T(A))$ comes from a self-adjoint element of A (see [BKR, Proposition 3.12] for instance), it extends to a real valued affine continuous function on $QT(A)$. Hence $D(K_0(A))$ is uniformly dense in $\mathrm{Aff}(T(A))$. Note that if one uses the deep result obtained by Haagerup in [H], the latter half of the proof is superfluous.

(3)$\Rightarrow$(1). Suppose that $\alpha \times \varphi$ is not rigid. Thus, there are $\nu_1, \nu_2 \in M_{\alpha \times \varphi}$ such that $F_*(\nu_1) = F_*(\nu_2) = \nu \in M_\alpha$, where F is the canonical factor map onto (X, α). Let τ_1 and τ_2 be the tracial states on A arising from ν_1 and ν_2. Since

$$\tau_{1*}([f]) = \nu_1(f \circ F) = \nu(f) = \nu_2(f \circ F) = \tau_{2*}([f])$$

for all $[f] \in K^0(X, \alpha)$, projections in $C^*(X, \alpha)$ cannot separate traces on A. Therefore, we can finish the proof here when $\alpha \times \varphi$ is not orientation preserving.

Assume that $\alpha \times \varphi$ is orientation preserving. By Lemma 2.4, $K_0(A)$ is isomorphic to $\mathbb{Z} \oplus K^0(X, \alpha)$. Suppose that there are projections $e_1, e_2 \in M_k(A)$ for some integer k such that $[e_1] - [e_2] = (1, 0) \in \mathbb{Z} \oplus K^0(X, \alpha)$. Then, for any $x = (n, [f]) \in K_0(A) \cong \mathbb{Z} \oplus K^0(X, \alpha)$, we have

$$\tau_{1*}(x) - \tau_{2*}(x) = n(\tau_1(e_1 - e_2) - \tau_2(e_1 - e_2)).$$

Hence

$$\{\tau_{1*}(x) - \tau_{2*}(x) : x \in K_0(A)\}$$

is discrete in $\mathbb{R}$, which is a contradiction. $\qquad\square$

We now turn to a consideration of stable rank of A. Suppose that $\alpha \times \varphi$ is minimal but may not be rigid. The C^*-subalgebra A_x may not have real rank zero. But, it is still a unital simple AT algebra by Proposition 3.3. Moreover, Lemma 3.7 is also valid when one replaces 'almost tracially AF' by 'almost AT'. A unital simple AT algebra is known to have property (SP), that is, every nonzero hereditary subalgebra contains a nonzero projection. Hence we see that A also has property (SP) by virtue of Lemma 3.7. We also remark that A_x has stable rank one and the order on projections of A_x is determined by traces, because A_x is a unital simple AT algebra. Then, by reading the proof of [Ph2, Theorem 5.2] carefully, it turns out that A and the "large" subalgebra A_x do not need to have real rank zero and that they only need to have property (SP) so that the proof works. As a consequence, we have the following.

Theorem 3.12. *When $\alpha \times \varphi$ is a minimal homeomorphism on $X \times \mathbb{T}$, the crossed product C^*-algebra $A = C^*(X \times \mathbb{T}, \alpha \times \varphi)$ has stable rank one. In particular, the projections in $A \otimes \mathcal{K}$ satisfy cancellation.*

In [Ph1, Theorem 4.5], it was proved that A satisfies the K-theoretic version of Blackadar's second fundamental comparability question, that is, if $x \in K_0(A)$ satisfies $\tau_*(x) > 0$ for all $\tau \in T(A)$, then $x \in K_0(A)^+$. In particular, $K_0(A)$ is weakly unperforated. Combining this with the theorem above, we can deduce the following.

Theorem 3.13. *When $\alpha \times \varphi$ is a minimal homeomorphism on $X \times \mathbb{T}$, the order on projections of $M_\infty(C^*(X \times \mathbb{T}, \alpha \times \varphi))$ is determined by traces. In other words, $C^*(X \times \mathbb{T}, \alpha \times \varphi)$ satisfies Blackadar's second fundamental comparability question.*

4   Cocycles with values in the rotation group

Let (X, α) be a Cantor minimal system and let $\xi : X \to \mathbb{T}$ be a continuous map. In this section, we would like to investigate a homeomorphism $\alpha \times R_\xi$ on $X \times \mathbb{T}$ and its related crossed product C^*-algebra $A = C^*(X \times \mathbb{T}, \alpha \times R_\xi)$. Of course, $\alpha \times R_\xi$ is orientation preserving.

Definition 4.1. Let α be a minimal homeomorphism on X. Define

$$K_\mathbb{T}^0(X, \alpha) \;=\; C(X, \mathbb{T}) / \{\eta - \eta\alpha^{-1} : \eta \in C(X, \mathbb{T})\}.$$

The equivalence class of $\xi \in C(X, \mathbb{T})$ in $K_\mathbb{T}^0(X, \alpha)$ is denoted by $[\xi]_\alpha$ or $[\xi]$.

Let $\theta \in \mathbb{T}$ and put $\xi(x) = \theta$ for all $x \in X$. Thus, ξ is a constant function. It is easy to see that $[\xi]$ is zero in $K_\mathbb{T}^0(X, \alpha)$ if and only if θ is a topological eigenvalue of (X, α). The reader may refer to [W, Theorem 5.17] for topological eigenvalues. Since X is compact, the set of topological eigenvalues is at most countable. It follows that $K_\mathbb{T}^0(X, \alpha)$ is uncountable.

At first, we describe when $\alpha \times R_\xi$ is minimal in terms of $K_\mathbb{T}^0(X, \alpha)$. Note that more general results were obtained in [Pa].

Lemma 4.2. *Let (X, α) be a Cantor minimal system and $\xi : X \to \mathbb{T}$ be a continuous map. Then, $\alpha \times R_\xi$ is minimal if and only if $n[\xi] \neq 0$ in $K_\mathbb{T}^0(X, \alpha)$ for all $n \in \mathbb{N}$.*

Proof. Suppose that there exist $n \in \mathbb{N}$ and η such that $n\xi = \eta - \eta\alpha^{-1}$. Then

$$\{(x, t) \in X \times \mathbb{T} : nt = \eta(\alpha^{-1}(x))\}$$

is closed and invariant under $\alpha \times R_\xi$, and so $\alpha \times R_\xi$ is not minimal.

Let us prove the other implication. Assume that $\alpha \times R_\xi$ is not minimal. Let E be a minimal subset of $\alpha \times R_\xi$. Note that $\mathrm{id} \times R_t$ commutes with $\alpha \times R_\xi$. Since E is not the whole of $X \times \mathbb{T}$,

$$G = \{t \in \mathbb{T} : (\mathrm{id} \times R_t)(E) = E\}$$

is a closed proper subgroup of $\mathbb{T}$. It follows that there exists $n \in \mathbb{N}$ such that $G = \{t \in \mathbb{T} : nt = 0\}$. Moreover, on account of the minimality of E, we can deduce that there exists $\eta : X \to \mathbb{T}$ such that

$$E = \{(x, t) \in X \times \mathbb{T} : nt = \eta(x)\}.$$

The map η is continuous, because E is closed. Hence we have $n\xi = \eta\alpha - \eta$. $\square$

If $\alpha \times R_\xi$ is not minimal, then there exist uncountably many minimal closed subsets. In particular, it is not rigid. Compare this with Remark 3.2.

Lemma 4.3. *Let (X, α) and (Y, β) be Cantor minimal systems. Let ξ and ζ be continuous maps from X to $\mathbb{T}$. Suppose that $\alpha \times R_\xi$ is minimal. Then, $\alpha \times \xi$ and $\beta \times \zeta$ is conjugate if and only if there exists a homeomorphism $F : X \to Y$ such that $F\alpha = \beta F$, and $[\xi]_\alpha = [\zeta F]_\alpha$ or $[\xi]_\alpha = -[\zeta F]_\alpha$ in $K_\mathbb{T}^0(X, \alpha)$.*

Proof. The 'if' part is clear. We consider the 'only if' part. Let $F \times \varphi : X \times \mathbb{T} \to Y \times \mathbb{T}$ be a conjugating map, that is,

$$F\alpha = \beta F \quad \text{and} \quad \varphi_{\alpha(x)}(s + \xi(x)) = \varphi_x(s) + \zeta(F(x))$$

for all $(x, s) \in X \times \mathbb{T}$. For every $t \in \mathbb{T}$, $\mathrm{id} \times R_t$ commutes with $\beta \times R_\zeta$, and so $(F \times \varphi)^{-1}(\mathrm{id} \times R_t)(F \times \varphi)$ commutes with $\alpha \times R_\xi$. Let $x \in X$ and put $s = \varphi_x^{-1}(\varphi_x(0) + t)$. Then we have

$$(F \times \varphi)^{-1}(\mathrm{id} \times R_t)(F \times \varphi)(x, 0) = (x, s) = (\mathrm{id} \times R_s)(x, 0).$$

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By the minimality of $\alpha \times \xi$, we can conclude that

$$(F \times \varphi)^{-1}(\mathrm{id} \times R_t)(F \times \varphi) = \mathrm{id} \times R_s.$$

It follows that the mapping $t \mapsto s$ is a continuous injective homomorphism from $\mathbb{T}$ to $\mathbb{T}$. Thus,

$$(F \times \varphi)^{-1}(\mathrm{id} \times R_t)(F \times \varphi) = \mathrm{id} \times R_t$$

for all $t \in \mathbb{T}$, or

$$(F \times \varphi)^{-1}(\mathrm{id} \times R_t)(F \times \varphi) = \mathrm{id} \times R_{-t}$$

for all $t \in \mathbb{T}$. Without loss of generality we may assume the first, which yields

$$\varphi_x(s + t) = \varphi_x(s) + t$$

for all $(x, s) \in X \times \mathbb{T}$ and $t \in \mathbb{T}$. It follows that φ_x equals $R_{\varphi_x(0)}$ and

$$\xi(x) + \varphi_{\alpha(x)}(0) = \varphi_x(0) + \zeta(F(x)).$$

Thereby the assertion follows. $\qquad\square$

Next, we would like to consider when $\alpha \times R_\xi$ is rigid. Although the following is a special case of [Pa, Theorem 3] or [Z, Theorem 3.5], we include the proof for the reader's convenience.

Lemma 4.4. *Let (X, α) be a Cantor minimal system. For a continuous function $\xi : X \to \mathbb{T}$, the following are equivalent.*

(1) *$\alpha \times R_\xi$ is rigid.*

(2) *Every $\alpha \times R_\xi$-invariant measure ν is also $\mathrm{id} \times R_t$-invariant for all $t \in \mathbb{T}$, that is, ν is a product measure of the Haar measure on $\mathbb{T}$ and an invariant measure for (X, α).*

(3) *For every α-invariant measure μ on X and $n \in \mathbb{N}$, there does not exist a Borel function $\eta : X \to \mathbb{T}$ such that $n\xi(x) = \eta(x) - \eta\alpha^{-1}(x)$ for μ-almost every $x \in X$.*

Proof. We denote the canonical factor map from $(X \times \mathbb{T}, \alpha \times R_\xi)$ to (X, α) by π.

(1)$\Rightarrow$(2). This is immediate from $\pi_* \circ (\mathrm{id} \times R_t)_* = (\pi \circ (\mathrm{id} \times R_t))_* = \pi_*$.

(2)$\Rightarrow$(1). Define a continuous map $\Phi : C(X \times \mathbb{T}) \to C(X)$ by

$$\Phi(f)(x) = \int_\mathbb{T} f(x, t)\, dt.$$

If $\nu \in M_{\alpha \times R_\xi}$ is $\mathrm{id} \times R_t$-invariant, then $\nu(f) = \pi_*(\nu)(\Phi(f))$ for all $f \in C(X \times \mathbb{T})$. Hence $\pi_*^{-1}(\pi_*(\nu)) = \{\nu\}$.

(2)$\Rightarrow$(3). Suppose that there exist an α-invariant measure $\mu \in M_\alpha$, $n \in \mathbb{N}$ and a Borel function $\eta : X \to \mathbb{T}$ such that

$$n\xi(x) = \eta(x) - \eta\alpha^{-1}(x)$$

for μ-almost every $x \in X$. Then

$$C(X \times \mathbb{T}) \ni f \mapsto \frac{1}{n} \int_X \sum_{nt = \eta\alpha^{-1}(x)} f(x, t)\, d\mu(x) \in \mathbb{C}$$

yields a probability measure on $X \times \mathbb{T}$. Note that the summation runs over n distinct t's which satisfy $nt = \eta\alpha^{-1}(x)$. This measure is $\alpha \times R_\xi$-invariant, because we have

$$\sum_{nt = \eta\alpha^{-1}(x)} f(\alpha(x), t + \xi(x)) = \sum_{nt = \eta\alpha^{-1}(\alpha(x))} f(\alpha(x), t)$$

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for μ-almost every $x \in X$. But it is not the product of the Haar measure and μ.

(3)$\Rightarrow$(2). Suppose that $\nu \in M_{\alpha \times R_\xi}$ is not invariant under the rotation $\mathrm{id} \times R_t$. We may assume that ν is an ergodic measure. It follows from [KH, Corollary 4.1.9] or [W, Lemma 6.13] that there exists an F_σ subset $E \subset X \times \mathbb{T}$ such that $\nu(E) = 1$ and

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f((\alpha \times R_\xi)^k(x, s)) = \nu(f)$$

for all $f \in C(X \times \mathbb{T})$ and $(x, s) \in E$. We may assume that E is $\alpha \times R_\xi$-invariant. Put

$$G = \{t \in \mathbb{T} : (\mathrm{id} \times R_t)_*(\nu) = \nu\}.$$

By assumption, G is a closed proper subgroup of $\mathbb{T}$. Thus, there is $n \in \mathbb{N}$ such that $G = \{t \in \mathbb{T} : nt = 0\}$. If (x, s) belongs to E, then we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f((\alpha \times R_\xi)^k(x, s+t)) = \nu(f \circ (\mathrm{id} \times R_t)) = (\mathrm{id} \times R_t)_*(\nu)(f)$$

for all $f \in C(X \times \mathbb{T})$. Therefore $(\mathrm{id} \times R_t)(E) \cap E$ is empty if $t \notin G$. Furthermore, by replacing E by

$$\bigcup_{t \in G} (\mathrm{id} \times R_t)(E),$$

we may assume that $(\mathrm{id} \times R_t)(E) = E$ for all $t \in G$. On the F_σ subset $\pi(E) \subset X$, we define a $\mathbb{T}$-valued function η by $\eta(x) = nt$ for $(x, t) \in E$. If $E_0 \subset E$ is closed, then η is evidently continuous on $\pi(E_0)$. It follows that η is a well-defined Borel function. For $(x, s) \in E$,

$$(\alpha \times R_\xi)(x, s) = (\alpha(x), s + \xi(x))$$

belongs to E, and so

$$\eta(\alpha(x)) = ns + n\xi(x) = \alpha(x) + n\xi(x)$$

is obtained. Since this equation holds for all $x \in \pi(E)$ and $\pi_*(\nu)(\pi(E)) \geq \nu(E) = 1$, the proof is completed. $\square$

Let $\mu \in M_\alpha$. As in the discussion following Definition 4.1, let $\xi(x) = \theta$ be a constant function. Then, there exists a Borel function $\eta \in C(X, \mathbb{T})$ such that

$$\xi(x) = \eta(x) - \eta\alpha^{-1}(x)$$

for μ-almost every $x \in X$ if and only if $e^{2\pi\sqrt{-1}\theta}$ is an eigenvalue of the unitary operator $\pi_\mu(u_\alpha) \in L^2(X, \mu)$, where π_μ is a representation of $C^*(X, \alpha)$ corresponding to the invariant measure μ. Since $L^2(X, \mu)$ is separable, eigenvalues of a unitary operator are at most countable. Hence, by the lemma above, we can obtain a lot of rigid homeomorphisms. Moreover, it is known that eigenvalues of the unitary operator $\pi_\mu(u_\alpha)$ need not be topological eigenvalues of (X, α). Therefore we can see that there exists a minimal homeomorphism $\alpha \times R_\xi$ which is not rigid. We will look at a concrete example in Example 9.1.

There is another way to find a rigid homeomorphism. Let $\xi \in C(X, \mathbb{T})$ and let $\tilde{\xi} \in C(X, \mathbb{R})$ be its lift (this is always possible because X is the Cantor set). Then $\mu \mapsto \mu(\tilde{\xi})$ gives an affine function from the set of α-invariant measures M_α to $\mathbb{R}$. By the lemma above, if $n\mu(\tilde{\xi}) \notin \mu(C(X, \mathbb{Z}))$ for each ergodic α-invariant measure μ and $n \in \mathbb{N}$, then $\alpha \times R_\xi$ is rigid.

Next, we will show that $A = C^*(X \times \mathbb{T}, \alpha \times R_\xi)$ can be written as a crossed product of $C^*(X, \alpha)$ by a certain action. Define an automorphism $\iota(\xi)$ on $C^*(X, \alpha)$ by $\iota(\xi)(f) = f$ for all

$f \in C(X)$ and $\iota(\xi)(u_\alpha) = u_\alpha e^{2\pi\sqrt{-1}\xi(x)}$, where u_α denotes the implementing unitary of $C^*(X, \alpha)$. This kind of automorphism was considered in [M1]. We remark that $\iota(\xi)$ is approximately inner, because $\iota(\xi)_*$ is the identity on the K-groups (or one can deduce it from Lemma 6.1 or [M1, Lemma 5.1]). Let $\hat\iota(\xi)$ denote the dual action on $C^*(X, \alpha) \rtimes_{\iota(\xi)} \mathbb{Z}$.

Proposition 4.5. *There is an isomorphism π from the crossed product C^*-algebra $A = C^*(X \times \mathbb{T}, \alpha \times R_\xi)$ to $C^*(X, \alpha) \rtimes_{\iota(\xi)} \mathbb{Z}$ such that $\pi(f) = f$ for all $f \in C(X)$ and $\pi(g \circ (\mathrm{id} \times R_t)) = \hat\iota(\xi)_t(\pi(g))$ for all $g \in C(X \times \mathbb{T})$ and $t \in \mathbb{T}$.*

Proof. In order to avoid confusion, we have to use different symbols for three implementing unitaries: we denote the implementing unitary in $C^*(X, \alpha)$ by u_α and denote the unitary implementing $\iota(\xi)$ by v, while $u \in A$ denotes the unitary implementing $\alpha \times R_\xi$.

Let $z \in C(X \times \mathbb{T})$ be a unitary defined by $z(x, t) = e^{2\pi\sqrt{-1}t}$. Define $\pi(z) = v$ and $\pi(f) = f$ for all $f \in C(X) \subset C(X \times \mathbb{T})$. This is well-defined because v and f commute in $C^*(X, \alpha) \rtimes \mathbb{Z}$. Moreover, π is an isomorphism from $C(X \times \mathbb{T})$ onto its image. We define $\pi(u) = u_\alpha$. It is easy to check that

$$\pi(u)\pi(f)\pi(u^*) = f\alpha^{-1} = \pi(f\alpha^{-1})$$

for all $f \in C(X)$ and that

$$\pi(u)\pi(z)\pi(u^*) = e^{-2\pi\sqrt{-1}\xi(\alpha^{-1}(x))}v = \pi(e^{-2\pi\sqrt{-1}\xi(\alpha^{-1}(x))}z) = \pi(z \circ \alpha^{-1}).$$

Therefore π is a homomorphism from A to $C^*(X, \alpha) \rtimes \mathbb{Z}$. Clearly π is surjective. It is also straightforward to see $\pi(g \circ (\mathrm{id} \times R_t)^{-1}) = \hat\iota(\xi)_t(\pi(g))$ for all $g \in C(X \times \mathbb{T})$ and $t \in \mathbb{T}$.

It remains to show that π is an isomorphism. Let E be the conditional expectation from A to $C(X \times \mathbb{T})$. It is well-known that E is faithful. We can define an action of $\mathbb{T}$ on $C^*(X, \alpha) \times \mathbb{Z}$ by

$$\gamma_t(f) = f, \quad \gamma_t(u_\alpha) = e^{2\pi\sqrt{-1}t}u_\alpha, \quad \text{and} \quad \gamma_t(v) = v$$

for $t \in \mathbb{T}$. Let

$$E_0(a) = \int_\mathbb{T} \gamma_t(a)\, dt$$

for $a \in C^*(X, \alpha) \rtimes \mathbb{Z}$. Then we have $\pi \circ E = E_0 \circ \pi$. The faithfulness of E leads us to the conclusion. $\square$

Remark 4.6. By [Pu1, Corollary 5.7] or [HPS, Theorem 5.5] there exist bijective correspondences between the following spaces.

(1) The state space $S(K^0(X, \alpha))$ of $K^0(X, \alpha)$.

(2) The tracial state space $T(C^*(X, \alpha))$ of $C^*(X, \alpha)$.

(3) The set M_α of all α-invariant probability measures.

By Proposition 4.5 and Lemma 4.4, (2) and (3) of the above are also identified with the following.

(4) The set of $\hat\iota(\xi)$-invariant traces on $A = C^*(X \times \mathbb{T}, \alpha \times R_\xi)$.

(5) The set of probability measures on $X \times \mathbb{T}$ which are invariant under $\alpha \times R_\xi$ and the rotation $\mathrm{id} \times R_t$.

Suppose that $s \in S(K_0(A))$ is a state on the ordered group $K_0(A)$. Then there is $\nu \in M_\alpha$ such that $s([f]) = S_\nu([f])$ for all $[f] \in K^0(X, \alpha)$, where S_ν is a state on $K^0(X, \alpha)$ coming from ν and $K^0(X, \alpha)$ is viewed as a subgroup of $K_0(A)$. The α-invariant measure ν extends to an $\alpha \times R_\xi$-invariant measure on $X \times \mathbb{T}$, and so we can extend S_ν on $K_0(A)$ (different choices of $\alpha \times R_\xi$-invariant measures do not concern the extension of S_ν). For $x \in K_0(A)$,

$$S(K^0(X, \alpha)) \ni S_\mu \mapsto S_\mu(x) \in \mathbb{R}$$

is an affine function on the state space $S(K^0(X, \alpha))$. Since the image of $K^0(X, \alpha)$ is dense in $\mathrm{Aff}(S(K^0(X, \alpha)))$, for any $\varepsilon > 0$, there exists $f_1, f_2 \in C(X, \mathbb{Z})$ such that

$$S_\mu(x) - \varepsilon < S_\mu([f_1]) < S_\mu(x) < S_\mu([f_2]) < S_\mu(x) + \varepsilon$$

for all $\mu \in M_\alpha$. If A is simple, then it follows from Theorem 3.13 that $[f_1] < x < [f_2]$ in $K_0(A)$. In particular, we have

$$S_\nu(x) - \varepsilon < S_\nu([f_1]) = s([f_1]) < s(x) < s([f_2]) = S_\nu([f_2]) < S_\nu(x) + \varepsilon.$$

Hence s is equal to S_ν as a state on $K_0(A)$. Consequently, the state space $S(K^0(X, \alpha))$ can be identified with

(6) The state space $S(K_0(A))$ of A

when A is simple.

Theorem 4.7. *Let (X, α) be a Cantor minimal system and let $\xi \in C(X, \mathbb{T})$. Suppose that $\alpha \times R_\xi$ is a minimal homeomorphism on $X \times \mathbb{T}$. For the unital simple C^*-algebra $A = C^*(X \times \mathbb{T}, \alpha \times R_\xi)$, the following conditions are equivalent.*

(1) *$\alpha \times R_\xi$ is rigid.*

(2) *A has real rank zero.*

(3) *For every extremal tracial state τ on $C^*(X, \alpha)$ and every $n \in \mathbb{N}$, $\iota(\xi)^n$ is not weakly inner in the GNS representation π_τ.*

If $C^*(X, \alpha)$ has finitely many extremal traces, then the conditions above are also equivalent to

(4) *$\iota(\xi)$ has the tracial cyclic Rohlin property.*

Proof. (1)$\Leftrightarrow$(2) was shown in Corollary 3.11.

(1)$\Rightarrow$(3) follows [K, Proposition 2.3].

(3)$\Rightarrow$(1). Suppose that $\alpha \times R_\xi$ is not rigid. By Lemma 4.4, there exist an ergodic measure $\mu \in M_\alpha$, $n \in \mathbb{N}$ and a Borel function $\eta : X \to \mathbb{T}$ such that

$$n\xi(x) = \eta(x) - \eta\alpha(x)$$

for μ-almost every $x \in X$. Define $h \in L^\infty(X, \mu)$ by $h(x) = e^{2\pi\sqrt{-1}\eta(x)}$ and let V be the multiplication operator by h on $L^2(X, \mu)$. Let τ be the extremal trace on $C^*(X, \alpha)$ corresponding to μ. We can regard $\pi_\tau(C^*(X, \alpha))$ as a C^*-subalgebra of $B(L^2(X, \mu))$. Then, it is not hard to see that V commutes with $\pi_\tau(f)$ for all $f \in C(X)$ and that

$$V^*\pi_\tau(u_\alpha)V = \pi_\tau(\iota(\xi)^n(u_\alpha)).$$

Namely $\iota(\xi)^n$ is weakly inner in the GNS representation π_τ.

(3)$\Rightarrow$(4). From [OP] we can see that $\iota(\xi)$ has the tracial Rohlin property. The conclusion follows from [LO, Theorem 3.4].

(4)$\Rightarrow$(2). It follows from [LO, Theorem 2.9] that A has tracial rank zero. The conclusion is immediate from [L1, Theorem 3.4]. $\square$

In the theorem above, it was shown that if $\alpha \times R_\xi$ is rigid and α has only finitely many ergodic measures, then A has tracial rank zero. In the next section we will prove that the hypothesis of finitely many ergodic measures is actually not necessary.

5 Tracial rank

Throughout this section, let (X, α) be a Cantor minimal system and let $\xi : X \to \mathbb{T}$ be a continuous map. We denote the crossed product C^*-algebra $C^*(X \times \mathbb{T}, \alpha \times R_\xi)$ by A and its implementing unitary by u. We would like to show that if $\alpha \times R_\xi$ is rigid then A has tracial rank zero. The proof will be done by some improvement of Lemma 3.7. Following the notation used there, we define $A_x = C^*(C(X \times \mathbb{T}), uC_0((X \setminus \{x\}) \times \mathbb{T}))$ for $x \in X$.

Define $z \in C(X \times \mathbb{T})$ by $z(x, t) = e^{2\pi\sqrt{-1}t}$. The key step of the proof is approximately unitary equivalence of $z1_U$ and $z1_V$ in A_x, where U and V are suitable clopen subsets of X satisfying $[1_U] = [1_V]$. When one uses the fact that A_x has tracial rank zero, the proof is just an application of [L6, Theorem 3.4], in which actually more general result has been obtained. But we would like to include an elementary proof which does not use tracial rank zero for the reader's convenience.

The following lemma says that rigidity implies that the values of a cocycle are uniformly distributed in $\mathbb{T}$.

Lemma 5.1. *Suppose that $\alpha \times R_\xi$ is rigid. For any irrational $s \in \mathbb{T}$ and any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that the following is satisfied: for any $n \geq N$ and $y \in X$ there is a permutation σ on $\{1, 2, \ldots, n\}$ such that*

$$|ks - \sum_{i=0}^{\sigma(k)-1} \xi(\alpha^i(y))| < \varepsilon$$

holds for all $k \in \{1, 2, \ldots, n\}$.

Proof. For $n \in \mathbb{N}$, put $\xi_n(x) = \sum_{i=0}^{n-1} \xi(\alpha^i(x))$. By Lemma 4.4, we have

$$\int_{X \times \mathbb{T}} f(t) \, d\nu = \int_\mathbb{T} f(t) \, dt$$

for every invariant measure ν of $\alpha \times R_\xi$ and every $f \in C(\mathbb{T})$. Hence for any $f \in C(\mathbb{T})$ and $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} f(\xi_n(x)) - \int_\mathbb{T} f(t) \, dt \right| < \varepsilon$$

for all $n \geq N$ and $x \in X$. By a slight modification of [KK, Lemma 2], we can get the conclusion. We leave the details to the reader. $\qquad\square$

In the following lemmas we need the idea of induced transformations. Let U be a clopen subset of X. Define $r : U \to \mathbb{N}$ by

$$r(x) = \min\{n \in \mathbb{N} : \alpha^n(x) \in U\}.$$

Since α is minimal, r is well-defined and continuous. Put $\tilde{\alpha}(y) = \alpha^{r(y)}(y)$ for every $y \in U$. Thus $\tilde{\alpha}$ is the first return map on U. It is well-known that $(U, \tilde{\alpha})$ is a Cantor minimal system and the associated crossed product $C^*(U, \tilde{\alpha})$ is canonically identified with $1_U C^*(X, \alpha) 1_U$. Define $\tilde{\xi} : U \to \mathbb{T}$ by

$$\tilde{\xi}(y) = \sum_{i=0}^{r(y)-1} \xi(\alpha^i(y))$$

for all $y \in U$. Then $\tilde{\alpha} \times R_{\tilde{\xi}}$ is the first return map of $\alpha \times R_\xi$ on $U \times \mathbb{T}$ and the associated crossed product $C^*(U \times \mathbb{T}, \tilde{\alpha} \times R_{\tilde{\xi}})$ is identified with $1_{U \times \mathbb{T}} A 1_{U \times \mathbb{T}}$. Note that the unitary implementing $\tilde{\alpha} \times R_{\tilde{\xi}}$ is given by

$$\sum_{n \in \mathbb{N}} u^n 1_{U_n \times \mathbb{T}},$$

where $U_n = r^{-1}(n)$ and the summation is actually finite.

In general there is a bijective correspondence between invariant measures of the induced transformation and those of the original one. It follows that $\alpha \times R_\xi$ is rigid if and only if $\tilde{\alpha} \times R_{\tilde{\xi}}$ is rigid.

For $x \in X$, let k be the minimal natural number such that $\alpha^{-k}(x) \in U$ and set $\tilde{x} = \alpha^{-k}(x)$. Then it is not hard to see that

$$1_U A_x 1_U = C^*(C(U \times \mathbb{T}), \tilde{u} C_0((U \setminus \{\tilde{x}\}) \times \mathbb{T})).$$

Lemma 5.2. *Let $\alpha \times R_\xi$ be a rigid homeomorphism and let $x \in X$. Suppose that U is a nonempty clopen subset of X. For any $s \in \mathbb{T}$ and any $\varepsilon > 0$, there exists a unitary $w \in 1_U(A_x \cap C^*(X, \alpha))1_U$ such that*

$$\|wzw^* - e^{2\pi\sqrt{-1}s} z 1_U\| < \varepsilon.$$

Proof. At first we consider the case $U = X$. Clearly we may assume that s is irrational. By applying Lemma 5.1 we can find $N \in \mathbb{N}$. Let

$$\mathcal{P} = \{X(v, k) : v \in V, k = 1, 2, \dots, h(v)\}$$

be a Kakutani-Rohlin partition such that the roof set $R(\mathcal{P})$ contains x and $h(v)$ is greater than N for every $v \in V$. By dividing each tower if necessary, we may assume that $\mathcal{P}$ is sufficiently finer so that whenever $y_1, y_2 \in X(v, k)$ we have $|\xi(y_1) - \xi(y_2)| < \varepsilon/h(v)$.

For every $v \in V$, choose $y_v \in X(v, 1)$ arbitrarily. By Lemma 5.1 there is a permutation σ_v on $\{1, 2, \dots, h(v)\}$ such that

$$\left|ks - \sum_{i=0}^{\sigma_v(k)-1} \xi(\alpha^i(y_v))\right| < \varepsilon$$

for all $k \in \{1, 2, \dots, h(v)\}$. Put

$$w = \sum_{v \in V} \sum_{i=1}^{h(v)} 1_{X(v, \sigma_v(i))} u^{\sigma_v(i) - \sigma_v(i+1)},$$

where u is the implementing unitary of $C^*(X, \alpha)$. It is easily verified that w is a unitary of $A_x \cap C^*(X, \alpha)$. Moreover we get the estimate

$$\|wzw^* - e^{2\pi\sqrt{-1}s} z\| < 4\pi\varepsilon.$$

Let us consider the general case. We follow the notation used in the discussion before the lemma. Applying the first part of the proof to $\tilde{\alpha} \times \tilde{\xi}$ and $\tilde{x}$, we obtain a unitary w in

$$C^*(C(U), \tilde{u} C_0(U \setminus \{\tilde{x}\})) = 1_U(A_x \cap C^*(X, \alpha))1_U$$

which satisfies the required inequality. $\square$

Lemma 5.3. *Let $\alpha \times R_\xi$ be a rigid homeomorphism and let $x \in X$. For any $\eta \in C(X, \mathbb{T})$ and any $\varepsilon > 0$, there exists a unitary $w \in A_x \cap C^*(X, \alpha)$ such that*

$$\|wzw^* - e^{2\pi\sqrt{-1}\eta}z\| < \varepsilon,$$

where $z \in C(X \times \mathbb{T})$ is given by $z(x, t) = e^{2\pi\sqrt{-1}t}$.

Proof. Let $\mathcal{P}$ be a partition of X such that whenever $y_1, y_2 \in U \in \mathcal{P}$ we have $|\eta(y_1) - \eta(y_2)| < \varepsilon$. For every $U \in \mathcal{P}$, by Lemma 5.2, we obtain a unitary $w_U \in 1_U(A_x \cap C^*(X, \alpha))1_U$ satisfying

$$\|w_U z w_U^* - e^{2\pi\sqrt{-1}\eta(\tilde{x})}z1_U\| < \varepsilon.$$

Let w be the product of all w_U's. Then w is the desired unitary. $\square$

Lemma 5.4. *Let $\alpha \times R_\xi$ be a rigid homeomorphism and let $x \in X$. Suppose that U is a clopen neighborhood of x and $U, \alpha(U), \ldots, \alpha^M(U)$ are mutually disjoint. Put $p = 1_U$ and $q = 1_{\alpha^M(U)}$. Then for any $\varepsilon > 0$ there exists a partial isometry $w \in A_x \cap C^*(X, \alpha)$ such that $w^*w = p$, $ww^* = q$ and*

$$\|wzw^* - zq\| < \varepsilon.$$

Moreover we have $u^{*i}wu^i \in A_x \cap C^*(X, \alpha)$ for all $i = 0, 1, \ldots, M - 1$.

Proof. There exists a partial isometry $v_1 \in A_x \cap C^*(X, \alpha)$ such that $v_1^*v_1 = p$ and $v_1v_1^* = q$. We have $v_1^*zv_1 = e^{2\pi\sqrt{-1}\eta}zp$ for some continuous function η defined on U. We consider the induced transformation on U. Let $\tilde{\alpha}$, $\tilde{\xi}$ and $\tilde{x}$ be as in the discussion before Lemma 5.2. Then Lemma 5.3 applies to them and yields a unitary $v_2 \in p(A_x \cap C^*(X, \alpha))p$ satisfying

$$\|v_2zpv_2^* - e^{2\pi\sqrt{-1}\eta}zp\| < \varepsilon.$$

Then $w = v_1v_2$ satisfies

$$\|wzw^* - zq\| < \varepsilon.$$

Since $U, \alpha(U), \ldots, \alpha^M(U)$ are mutually disjoint, one can check that w belongs to $A_{\alpha^i(x)} \cap C^*(X, \alpha)$ for all $i = 0, 1, \ldots, M - 1$. It follows that $u^{*i}wu^i \in A_x \cap C^*(X, \alpha)$ for all $i = 0, 1, \ldots, M - 1$. $\square$

Lemma 5.5. *Suppose that $\alpha \times R_\xi$ is rigid. Let $x \in X$. For any $N \in \mathbb{N}$, $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset C(X \times \mathbb{T})$, we can find a natural number $M > N$, a clopen neighborhood U of x and a partial isometry $w \in A_x$ which satisfy the following.*

(1) *$\alpha^{-N+1}(U), \alpha^{-N+2}(U), \ldots, U, \alpha(U), \ldots, \alpha^M(U)$ are mutually disjoint, and $\mu(U) < \varepsilon/M$ for all α-invariant measure μ.*

(2) *$w^*w = 1_U$ and $ww^* = 1_{\alpha^M(U)}$.*

(3) *$u^{*i}wu^i \in A_x$ for all $i = 0, 1, \ldots, M - 1$.*

(4) *$\|wf - fw\| < \varepsilon$ for all $f \in \mathcal{F}$.*

Proof. Without loss of generality, we may assume $\mathcal{F} = \{f_1, f_2, \ldots, f_k, z\}$, where f_i belongs to $C(X) \subset C(X \times \mathbb{T})$. There exists a clopen neighborhood O of x such that

$$|f_i(x) - f_i(y)| < \varepsilon/2$$

for all $y \in O$ and $i = 1, 2, \ldots, k$. Since α is minimal, we can find $M > N$ such that $\alpha^M(x) \in O$. Let U be a clopen neighborhood of x such that the condition (1) is satisfied and $U \cup \alpha^M(U) \subset O$. Now Lemma 5.4 applies and yields a partial isometry w. It is clear that w is the desired one. $\square$

Theorem 5.6. *Suppose that $\alpha \times R_\xi$ is minimal. Then the following are equivalent.*

(1) $\alpha \times R_\xi$ *is rigid.*

(2) $A = C(X \times \mathbb{T}, \alpha \times R_\xi)$ *has real rank zero.*

(3) $A = C(X \times \mathbb{T}, \alpha \times R_\xi)$ *has tracial rank zero.*

(4) $A = C(X \times \mathbb{T}, \alpha \times R_\xi)$ *is a unital simple AT-algebra with real rank zero.*

Proof. It has been proved in Corollary 3.11 that (1) and (2) are equivalent.

(3)$\Leftrightarrow$(4) follows the classification theorem of [L5], since A has torsion free K-theory (see Lemma 2.4).

(4)$\Rightarrow$(2) is obvious.

(1)$\Rightarrow$(3). We will show the following: For any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset C(X \times \mathbb{T})$ and any nonzero positive element $c \in A$, there exists a projection $e \in A_x$ such that the following conditions hold.

- $\|ae - ea\| < \varepsilon$ for all $a \in \mathcal{F} \cup \{u\}$.

- For any $a \in \mathcal{F} \cup \{u\}$, there exists $b \in eA_xe$ such that $\|eae - b\| < \varepsilon$.

- $1 - e$ is equivalent to a projection in $\overline{cAc}$.

It follows from Proposition 3.3 that eA_xe is a unital simple AT-algebra with real rank zero. Therefore it has tracial rank zero (for example see [L3, Theorem 4.3.5]). Thus, if the above is proved, by [HLX, Theorem 4.8], A has tracial rank zero. In Section 3 it was proved that A has real rank zero, stable rank one and has weakly unperforated $K_0(A)$, and so it suffices to show the following: For any $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset C(X \times \mathbb{T})$, there exists a projection $e \in A_x$ such that the following conditions hold.

- $\|ae - ea\| < \varepsilon$ for all $a \in \mathcal{F} \cup \{u\}$.

- For any $a \in \mathcal{F} \cup \{u\}$, there exists $b \in eA_xe$ such that $\|eae - b\| < \varepsilon$.

- $\tau(1 - e) < \varepsilon$ for all $\tau \in T(A)$.

We may assume $\mathcal{F}^* = \mathcal{F}$. Choose $N \in \mathbb{N}$ so that $2\pi/N$ is less than ε. Applying Lemma 5.5 to N, $\varepsilon/2$ and a finite subset

$$\mathcal{G} = \bigcup_{i=0}^{N-1} u^i \mathcal{F} u^{i*},$$

we obtain $M > N$, a clopen neighborhood U of x and a partial isometry $w \in A_x$. Put $p = 1_U$ and $q = 1_{\alpha^M(U)}$. For $t \in [0, \pi]$ we define

$$P(t) = p\cos^2 t + w\sin t \cos t + w^* \sin t \cos t + q \sin^2 t.$$

Then $P(t)$ is a continuous path of projections with $P(0) = p$ and $P(\pi) = q$. By the choice of w we obtain the estimate

$$\|u^{i*}P(t)u^i f - f u^{i*}P(t)u^i\| < \varepsilon$$

for all $t \in [0, \pi]$, $i = 0, 1, \ldots, N-1$ and $f \in \mathcal{F}$. We define a projection e by

$$e = 1 - \left(\sum_{i=0}^{M-N} u^i p u^{i*} + \sum_{i=1}^{N-1} u^{i*} P(i\pi/N) u^i \right).$$

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The partial isometry w satisfies $u^{i*}wu^i \in A_x$ for all $i = 1, 2, \ldots, N-1$, and so e is a projection of A_x. Evidently we have

$$\|fe - ef\| < \varepsilon$$

for all $f \in \mathcal{F}$. Since

$$\|P(i\pi/N) - P((i-1)\pi/N)\| < \frac{2\pi}{N} < \varepsilon,$$

it is not hard to see

$$\|ue - eu\| < \varepsilon.$$

It is clear that efe belongs to A_x for all $f \in C(X \times \mathbb{T})$. It follows from $eue = eu(1-p)e$ that eue also belongs to A_x. We can easily verify

$$\tau(1 - e) < M\tau(p) < \frac{\varepsilon}{2}$$

for all $\tau \in T(A)$. $\square$

6 The generalized Rieffel projection

By Lemma 2.4, the K_i-group ($i = 1, 2$) of the crossed product C^*-algebra arising from an orientation preserving homeomorphism $\alpha \times \varphi$ is isomorphic to the direct sum of $\mathbb{Z}$ and $K^0(X, \alpha)$. Needless to say, the equivalence class of the implementing unitary is the generator of $\mathbb{Z}$ in the K_1-group. This section is devoted to specify a projection of $C^*(X \times \mathbb{T}, \alpha \times \varphi)$ which gives a representative of the generator of the $\mathbb{Z}$-summand of the K_0-group.

At first, we consider the case that a cocycle takes its values in the rotation group of the circle. Let (X, α) be a Cantor minimal system and let $\xi : X \to \mathbb{T}$ be a continuous map. We denote $C^*(X \times \mathbb{T}, \alpha \times R_\xi)$ by A for short and denote the implementing unitary by u. We will identify $K^0(X, \alpha)$ as a subgroup of $K_0(A)$.

Let $\tilde{\xi} \in C(X, \mathbb{R})$ be an arbitrary lift of $\xi \in C(X, \mathbb{T})$. Then $M_\alpha \ni \mu \mapsto \mu(\tilde{\xi}) \in \mathbb{R}$ gives an affine function on the set of invariant probability measures M_α. The other lifts of ξ are of the form $\tilde{\xi} + f$ with $f \in C(X, \mathbb{Z})$, and so this affine function is uniquely determined up to the natural image of $K^0(X, \alpha)$ in $\mathrm{Aff}(M_\alpha)$. Suppose $[\xi] = 0$ in $K^0_{\mathbb{T}}(X, \alpha)$. Then there exists $\eta \in C(X, \mathbb{T})$ such that $\xi = \eta - \eta\alpha^{-1}$. When $\tilde{\eta} \in C(X, \mathbb{R})$ is a lift of η, $\tilde{\xi} = \tilde{\eta} - \tilde{\eta}\alpha^{-1}$ is a lift of ξ and $\mu(\tilde{\xi}) = 0$ for all $\mu \in M_\alpha$. Therefore we obtain a homomorphism

$$\Delta : K^0_{\mathbb{T}}(X, \alpha) \ni [\xi] \;\mapsto\; \Delta([\xi]) \in \mathrm{Aff}(M_\alpha)/D(K^0(X, \alpha)).$$

This homomorphism Δ is known to be surjective ([M1, Lemma 6.2]).

Lemma 6.1. *Let (X, α) be a Cantor minimal system and let $\xi : X \to \mathbb{T}$ be a continuous map. For any $\varepsilon > 0$, there exists $\eta \in C(X, \mathbb{T})$ such that $|(\xi + \eta - \eta\alpha)(x)| < \varepsilon$ for all $x \in X$.*

Proof. Let

$$\mathcal{P} = \{X(v, k) : v \in V, k = 1, 2, \ldots, h(v)\}$$

be a Kakutani-Rohlin partition of (X, α) such that $h(v) > \varepsilon^{-1}$ for every $v \in V$. We denote the roof set $\bigcup_{v \in V} X(v, h(v))$ by $R(\mathcal{P})$. Put

$$\kappa(x) = \sum_{k=1}^{h(v)} \xi(\alpha^{k-1}(x))$$

for all $x \in X(v, 1)$ and $v \in V$. Since X is totally disconnected, there exists a real valued continuous function $\tilde{\kappa}$ on $\alpha(R(\mathcal{P}))$ such that

$$\tilde{\kappa}(x) + \mathbb{Z} = \kappa(x) \quad \text{and} \quad -1 < \tilde{\kappa}(x) < 1$$

for all $x \in \alpha(R(\mathcal{P})) = \bigcup_{v \in V} X(v, 1)$. Define $\eta \in C(X, \mathbb{T})$ by $\eta(x) = 0$ for all $x \in \alpha(R(\mathcal{P}))$ and

$$\eta(\alpha^k(x)) = \sum_{i=1}^{k} \xi(\alpha^{i-1}(x)) - \frac{k}{h(v)}\tilde{\kappa}(x) + \mathbb{Z}$$

for $x \in X(v, 1)$, $v \in V$ and $k = 1, 2, \ldots, h(v) - 1$. It is not hard to see that η is the desired function. $\qquad\square$

In a similar fashion to the lemma above, we can show the following, which will be used later.

Lemma 6.2. *Let (X, α) be a Cantor minimal system and let $\xi : X \to \mathbb{T}$ and $c : X \to \mathbb{Z}_2$ be continuous maps. For any $\varepsilon > 0$, there exists $\eta \in C(X, \mathbb{T})$ such that*

$$|\xi(x) + \eta(x) - (-1)^{c(x)}\eta\alpha(x)| < \varepsilon$$

for all $x \in X$.

Definition 6.3. *Let (X, α) be a Cantor minimal system and let $\xi : X \to \mathbb{T}$ be a continuous map. Define*

$$H(\alpha, \xi) = \{\eta \in C(X, \mathbb{T}) \ : \ (\xi + \eta - \eta\alpha)(x) \in (1/10, 9/10) \text{ for all } x \in X\}.$$

By Lemma 6.1, $H(\alpha, \xi)$ is not empty.

Suppose that η belongs to $H(\alpha, \xi)$. We define a projection $e(\alpha, \xi, \eta)$ in A as follows. Define a real valued continuous function g_η on $X \times \mathbb{T}$ by

$$g_\eta(x, t) = \begin{cases} \sqrt{10(t - \eta(x))(1 - 10(t - \eta(x)))} & t \in [\eta(x), \eta(x) + 1/10] \\ 0 & \text{otherwise.} \end{cases}$$

Put $\eta' = (\xi + \eta) \circ \alpha^{-1}$ and define a real valued continuous function $f(\alpha, \xi, \eta)$ on $X \times \mathbb{T}$ by

$$f(\alpha, \xi, \eta)(x, t) = \begin{cases} 10(t - \eta(x)) & t \in [\eta(x), \eta(x) + 1/10] \\ 1 & t \in [\eta(x) + 1/10, \eta'(x)] \\ 1 - 10(t - \eta'(x)) & t \in [\eta'(x), \eta'(x) + 1/10] \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to check that $e(\alpha, \xi, \eta) = g_\eta u^* + f(\alpha, \xi, \eta) + u g_\eta \in A$ is a self-adjoint projection. We call $e(\alpha, \xi, \eta)$ the generalized Rieffel projection. Let $\tilde{\xi} \in C(X, \mathbb{R})$ be a lift of $\xi + \eta - \eta\alpha$ such that $1/10 < \tilde{\xi} < 9/10$. Then, for every $\mu \in M_\alpha$,

$$\tau_\mu(e(\alpha, \xi, \eta)) = \tau_\mu(f(\alpha, \xi, \eta)) = \mu(\tilde{\xi}),$$

where τ_μ is the tracial state on A corresponding to μ. Hence the affine function $\mu \mapsto \tau_\mu(e(\alpha, \xi, \eta))$ is a representative of $\Delta([\xi]) \in \mathrm{Aff}(M_\alpha)/D(K^0(X, \alpha))$.

Proposition 6.4. *In the situation above, let $e = e(\alpha, \xi, \eta) \in A$ be the generalized Rieffel projection. Then $K^0(X, \alpha)$ and $[e]$ generate $K_0(A)$.*

Proof. Let v the unilateral shift on $\ell^2(\mathbb{N})$ and let $\mathcal{T}$ be the Toeplitz algebra generated by v. Put $q = 1 - vv^*$. In the C^*-algebra $A \otimes \mathcal{T}$, we consider the C^*-subalgebra B generated by $C(X \times \mathbb{T}) \otimes 1$ and $u \otimes v^*$. There is a surjective homomorphism π from B to A sending $u \otimes v^*$ to u. The kernel of π is $C(X \times \mathbb{T}) \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators. Put $a = g_\eta u^* \otimes v + f(\alpha, \xi, \eta) \otimes 1 + u g_\eta \otimes v^*$. Then $\pi(a) = e$. Hence it suffices to show that $e^{2\pi\sqrt{-1}a}$ is a generator of $K_1(\mathbb{C}1_X \otimes C(\mathbb{T})) \cong \mathbb{Z}$. Define a continuous function $h(t)$ by

$$h(t) = (1 - 8t(1-t)) + \sqrt{-1}t(t-1)(t-1/2).$$

Since $e^{2\pi\sqrt{-1}t}$ is homotopic to $h(t)$ in the set of complex-valued invertible functions, it suffices to know the K_1-class of the invertible element $h(a)$. By

$$a - a^2 = g_\eta^2 \otimes q$$

and

$$(a^2 - a)(a - 1/2) = -(g_\eta^2(f(\alpha, \xi, \eta) - 1/2)) \otimes q,$$

it follows that $h(a)$ is homotopic to $1_X \otimes z \otimes q$ in $GL(C(X \times \mathbb{T})) \otimes q$, where $z(t) = e^{2\pi\sqrt{-1}t}$ is the generator of $K_1(C(\mathbb{T}))$. $\square$

Let $K_0(A) \cong \mathbb{Z} \oplus K^0(X, \alpha)$ be the isomorphism described in the proposition above. If $\alpha \times R_\xi$ is minimal, then it follows from Theorem 3.13 that

$$K_0(A)^+ \cong \{(n, [f]) : \mu(n\tilde{\xi} + f) > 0 \text{ for all } \mu \in M_\alpha\} \cup \{0\}.$$

See also Remark 4.6.

We now turn to the general case. Let (X, α) be a Cantor minimal system and let $\varphi : X \to \text{Homeo}^+(\mathbb{T})$ be a continuous map. We write the crossed product C^*-algebra arising from $(X \times \mathbb{T}, \alpha \times \varphi)$ by A and the implementing unitary by u. In order to define the Rieffel projection in A, we need some preparations. For $\varphi \in \text{Homeo}^+(\mathbb{T})$, let $r(\varphi) \in \mathbb{T}$ denote the rotation number. The reader may refer to [KH, Chapter 11] for the definition and some elementary properties of $r(\varphi)$.

Lemma 6.5. *Let $\varphi \in \text{Homeo}^+(\mathbb{T})$. The map $\mathbb{T} \ni t \mapsto r(R_t\varphi)$ is a continuous surjection from $\mathbb{T}$ to $\mathbb{T}$ of degree one.*

Proof. By the definition of the rotation number, we see that the mapping $t \mapsto r(R_t\varphi)$ is nondecreasing as a real valued function. It is clear that $r(R_t\varphi) = 0$ if and only if t belongs to

$$I = \{s - \varphi(s) : s \in \mathbb{T}\}.$$

Since φ is an orientation preserving homeomorphism, I is not the whole circle. Thus, I is a closed interval of the circle. It follows that the map is a surjection of degree one. $\square$

Lemma 6.6. *Let X be the Cantor set and let $I \subset \mathbb{T}$ be an open subset. When $\Phi : X \times \mathbb{T} \to \mathbb{T}$ is a continuous map and $\Phi(x, \cdot)$ is surjective for every $x \in X$, there exists a continuous map $\xi : X \to \mathbb{T}$ such that $\Phi(x, \xi(x)) \in I$ for all $x \in X$.*

Proof. By assumption, for each $x \in X$, there exists $t_x \in \mathbb{T}$ such that $\Phi(x, t_x) \in I$. The continuity of Φ implies that there exists a clopen neighborhood U_x of x such that $\Phi(U_x, t_x) \subset I$. Since X is compact, it is covered by finitely many U_x's. We can find a locally constant function $\xi \in C(X, \mathbb{T})$ satisfying the required property. $\square$

The following lemma corresponds to Lemma 6.1.

Lemma 6.7. *Let (X, α) be a Cantor minimal system and let $\varphi : X \to \mathrm{Homeo}^+(\mathbb{T})$ be a continuous map. Suppose that an open subset $I \subset \mathbb{T}$ is given. Then there exists $\eta \in C(X, \mathbb{T})$ such that*

$$r(R_{\eta\alpha(x)} \circ \varphi_x \circ R_{\eta(x)}^{-1}) \in I$$

for all $x \in X$.

Proof. Define $\Phi(x, t) = r(R_t \varphi_x)$. It is obvious that Φ is continuous. By Lemma 6.5, for each $x \in X$, $\Phi(x, \cdot)$ is a continuous surjection. It follows from Lemma 6.6 that there exists a continuous map $\xi : X \to \mathbb{T}$ such that $\Phi(x, \xi(x)) \in I$ for all $x \in X$. Moreover, there exists $\varepsilon > 0$ such that, for all $x \in X$, if $|t - \xi(x)| < \varepsilon$ then $\Phi(x, t) \in I$. From Lemma 6.1, we can find $\eta \in C(X, \mathbb{T})$ such that

$$|(\xi + \eta - \eta\alpha)(x)| < \varepsilon$$

for all $x \in X$. Then we get

$$r(R_{\eta\alpha(x)} \circ \varphi_x \circ R_{\eta(x)}^{-1})$$
$$= r(R_{\eta(x)} \circ R_{\eta\alpha(x)-\eta(x)} \circ \varphi_x \circ R_{-\eta(x)})$$
$$= r(R_{\eta\alpha(x)-\eta(x)} \circ \varphi_x)$$
$$= \Phi(x, \eta\alpha(x) - \eta(x)) \in I,$$

thereby completing the proof. $\square$

By the lemma above, without loss of generality, we may always assume that $r(\varphi_x)$ is not zero for all $x \in X$. Put

$$c = \inf\{|\varphi_x(t) - t|, |\varphi_x^{-1}(t) - t| : (x, t) \in X \times \mathbb{T}\}.$$

Since φ_x has no fixed points, c is a positive real number. Take $s \in \mathbb{T}$. We define a function $f(\alpha, \varphi, s) \in C(X \times \mathbb{T})$ by

$$f(\alpha, \varphi, s)(x, t) = \begin{cases} c^{-1}(t - s) & t \in [s, s + c] \\ 1 & t \in [s + c, \varphi_{\alpha^{-1}(x)}(s)] \\ 1 - c^{-1}(\varphi_{\alpha^{-1}(x)}^{-1}(t) - s) & t \in [\varphi_{\alpha^{-1}(x)}(s), \varphi_{\alpha^{-1}(x)}(s + c)] \\ 0 & \text{otherwise.} \end{cases}$$

By the choice of c, f is well-defined. Define a function $g_s \in C(X \times \mathbb{T})$ by

$$g_s(x, t) = \begin{cases} \sqrt{c^{-1}(t - s)(1 - c^{-1}(t - s))} & t \in [s, s + c] \\ 0 & \text{otherwise.} \end{cases}$$

Then one checks that $e(\alpha, \varphi, s) = g_s u^* + f(\alpha, \varphi, s) + u g_s$ is a well-defined projection of A. Let us call it the generalized Rieffel projection for A. In exactly the same way as Proposition 6.4, we can show the following.

Proposition 6.8. *In the above setting, $K_0(A)$ is generated by $K^0(X, \alpha)$ and $[e(\alpha, \varphi, s)]$. Furthermore, $\mathbb{T} \ni s \mapsto e(\alpha, \varphi, s) \in A$ is a continuous path of projections in A.*

In the definition of $e(\alpha, \varphi, s)$, we can replace g_s by $z g_s$, where z is a complex number with $|z| = 1$. But, this choice does not matter to the homotopy equivalence class of the projection.

7   Approximate K-conjugacy

Let us begin with recalling the definition of weakly approximate conjugacy.

Definition 7.1 ([LM, Definition 3.1]). Let (X, α) and (Y, β) be dynamical systems on compact metrizable spaces X and Y. We say that (X, α) and (Y, β) are weakly approximately conjugate, if there exist homeomorphisms $\sigma_n : X \to Y$ and $\tau_n : Y \to X$ such that $\sigma_n \alpha \sigma_n^{-1}$ converges to β in $\mathrm{Homeo}(Y)$ and $\tau_n \beta \tau_n^{-1}$ converges to α in $\mathrm{Homeo}(X)$.

In [LM, Theorem 4.13], it was shown that two Cantor minimal systems are weakly approximately conjugate if and only if they have the same periodic spectrum (see also [M3, Theorem 3.1]). Similar results were shown in [M3] for dynamical systems on the product of the Cantor set and the circle.

As one sees that in the above definition, there is no consistency among σ_n or τ_n. It is clear (see [LM]) that the relation can be made stronger if one requires some consistency among σ_n as well as τ_n. We hope such stronger version of approximate conjugacy is more reasonable replacement of (flip) conjugacy.

Suppose $\sigma_n \alpha \sigma_n^{-1} \to \beta$ in $\mathrm{Homeo}(Y)$. In [LM, Proposition 3.2], it was shown that there exists an asymptotic morphism $\{\psi_n\} : B \to A$ such that

$$\lim_{n \to \infty} \|\psi_n(f) - f \circ \sigma_n\| = 0$$

for all $f \in C(Y)$ and

$$\lim_{n \to \infty} \psi_n(u_\beta) = u_\alpha,$$

where u_α and u_β denote the implementing unitaries in $C^*(X, \alpha)$ and $C^*(Y, \beta)$. This observation, however, is far from the existence theorem in classification theory, which requires that $\{\psi_n\}$ carries an isomorphism of K-groups (see [L5, Theorem 4.3] for instance). As pointed out in [LM], we have to impose conditions on the conjugating maps σ_n so that the associated asymptotic morphism has a nice property. Taking account of this, we make the following definitions. By an order and unit preserving homomorphism $\rho : K_*(B) \to K_*(A)$, we mean a pair of homomorphisms $\rho_i : K_i(A) \to K_i(B)$ $(i = 0, 1)$ such that $\rho_0([1_A]) = [1_B]$ and $\rho_0(K_0(A)^+) \subset K_0(B)^+$.

Definition 7.2. Let (X, α) and (Y, β) be dynamical systems on compact metrizable spaces X and Y. Suppose that a sequence of homeomorphisms $\sigma_n : X \to Y$ satisfies $\sigma_n \alpha \sigma_n^{-1} \to \beta$ in $\mathrm{Homeo}(Y)$. Let $\{\psi_n\}$ be the asymptotic morphism arising from σ_n. We say that the sequence $\{\sigma_n\}$ induces an order and unit preserving homomorphism $\rho : K_*(C^*(Y, \beta)) \to K_*(C^*(X, \alpha))$ between K-groups, if for every projection $p \in M_\infty(C^*(Y, \beta))$ and every unitary $u \in M_\infty(C^*(Y, \beta))$, there exists $N \in \mathbb{N}$ such that

$$[\psi_n(p)] = \rho([p]) \in K_0(C^*(X, \alpha)) \quad \text{and} \quad [\psi_n(u)] = \rho([u]) \in K_1(C^*(X, \alpha))$$

for every $n \geq N$.

Definition 7.3 ([L6, Definition 5.3]). Let (X, α) and (Y, β) be dynamical systems on compact metrizable spaces X and Y. We say that (X, α) and (Y, β) are approximately K-conjugate, if there exist homeomorphisms $\sigma_n : X \to Y$, $\tau_n : Y \to X$ and an isomorphism $\rho : K_*(C^*(Y, \beta)) \to K_*(C^*(X, \alpha))$ between K-groups such that

$$\sigma_n \alpha \sigma_n^{-1} \to \beta, \quad \tau_n \beta \tau_n^{-1} \to \alpha$$

and the associated asymptotic morphisms $\{\psi_n\} : B \to A$ and $\{\varphi_n\} : A \to B$ induce the isomorphisms ρ and ρ^{-1}.

We say that (X, α) and (Y, β) are approximately flip K-conjugate, if (X, α) is approximately K-conjugate to either of (Y, β) and (Y, β^{-1}).

J. Tomiyama [T] proved that two topological transitive systems (X, α) and (Y, β) are flip conjugate if and only if there is an isomorphism $\varphi : C^*(X, \alpha) \to C^*(Y, \beta)$ such that $\varphi \circ j_\alpha = j_\beta \circ \chi$ for some isomorphism $\chi : C(X) \to C(Y)$.

Definition 7.4 ([L6, Definition 3.8]). Let (X, α) and (Y, β) be two topological transitive systems. We say that (X, α) and (Y, β) are C^*-strongly approximately flip conjugate if there exists a sequence of isomorphisms $\varphi_n : C^*(X, \alpha) \to C^($Y, \beta)$ and a sequence of isomorphisms $\chi_n : C(X) \to C(Y)$ such that $[\varphi_n] = [\varphi_1]$ in $KL(C^*(X, \alpha), C^*(Y, \beta))$ for all $n \in \mathbb{N}$ and

$$\lim_{n \to \infty} \| \varphi_n \circ j_\alpha(f) - j_\beta \circ \chi_n(f) \| = 0$$

for all $f \in C(X)$.

Let X be the Cantor set and let ξ be a continuous function from X to $\mathbb{T}$. In this section, we would like to discuss approximate K-conjugacy for $(X \times \mathbb{T}, \alpha \times R_\xi)$.

As in the previous section, let A denote the crossed product C^*-algebra $C^*(X \times \mathbb{T}, \alpha \times R_\xi)$. By Lemma 6.1, we can find a continuous function $\zeta : X \to \mathbb{T}$ such that $[\zeta] = [\xi]$ in $K_\mathbb{T}^0(X, \alpha)$ and $\zeta(x) \in (7/15, 8/15)$ for every $x \in X$. By Lemma 4.3, $\alpha \times R_\xi$ is conjugate to $\alpha \times R_\zeta$. Therefore we may assume $\xi(x) \in (7/15, 8/15)$ for all $x \in X$ without loss of generality.

Suppose that η belongs to $H(\alpha, \xi)$. Let $\tilde{\eta} \in C(X, \mathbb{R})$ be a lift of η. Since $(\xi + \eta - \eta\alpha)(x) \in (1/10, 9/10)$ for every $x \in X$, we have $(\eta - \eta\alpha)(x) \notin [13/30, 17/30]$. Hence there exists a unique $f \in C(X, \mathbb{Z})$ such that

$$\| f - (\tilde{\eta} - \tilde{\eta}\alpha) \|_\infty < \frac{1}{2}.$$

It is easy to see that $[f] \in K^0(X, \alpha)$ is independent of the choice of $\tilde{\eta}$. Let us denote $[f]$ by $B_\alpha(\eta)$.

Lemma 7.5. *In the above setting, suppose that η and η' are homotopic in $H(\alpha, \xi)$.*

(1) *$e(\alpha, \xi, \eta)$ is homotopic to $e(\alpha, \xi, \eta')$ in the set of projections of A.*

(2) *$B_\alpha(\eta) = B_\alpha(\eta')$.*

Proof. (1) Suppose $[0, 1] \ni t \mapsto \kappa_t \in H(\alpha, \xi)$ is a homotopy from η to η'. Then the generalized Rieffel projection $e(\alpha, \xi, \kappa_t)$ is well-defined and $t \mapsto e(\alpha, \xi, \kappa_t)$ gives a continuous path of projections in A from $e(\alpha, \xi, \eta)$ to $e(\alpha, \xi, \eta')$.

(2) Let κ_t be a homotopy as in (1). There exists a continuous map $\tilde{\kappa}$ from $X \times [0, 1]$ to $\mathbb{R}$ such that

$$\tilde{\kappa}_t(x) + \mathbb{Z} = \kappa_t(x)$$

for all $x \in X$ and $t \in [0, 1]$. Since

$$(\tilde{\kappa}_t - \tilde{\kappa}_t\alpha)(x) \neq \frac{1}{2},$$

the integer nearest to $(\tilde{\kappa}_t - \tilde{\kappa}_t\alpha)(x)$ does not vary as t varies. Hence we get $B_\alpha(\eta) = B_\alpha(\eta')$. $\square$

Let $\eta \in H(\alpha, \xi)$. The unitary $v_\eta(x) = e^{2\pi\sqrt{-1}\eta(x)}$ of $C(X)$ satisfies $\| u_\alpha v_\eta u_\alpha^* v_\eta^* - 1 \|_\infty < 2$, where u_α denotes the implementing unitary of $C^*(X, \alpha)$, because of $(\eta - \eta\alpha)(x) \neq 1/2$. Thus, u_α and v_η are almost commuting unitaries in a sense. When $u, v \in M_n(\mathbb{C})$ are unitaries satisfying $\| uv - vu \| < 2$, on account of $\det(uvu^*v^*) = 1$, we have

$$\frac{1}{2\pi\sqrt{-1}} \mathrm{Tr}(\log(uvu^*v^*)) \in \mathbb{Z} \cong K_0(M_n(\mathbb{C})),$$

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where Tr is the standard trace on $M_n(\mathbb{C})$ and log is the logarithm with values in $\{z : \Im(z) \in (-\pi, \pi)\}$. The Bott element for pairs of almost commuting unitaries in a unital C^*-algebra is a generalization of this (see [EL]). More precisely, if u and v are unitaries in a unital C^*-algebra and $\|uv - vu\| \approx 0$, then a projection $B(u, v)$ and an element of the K_0-group are obtained. Our $B_\alpha(\eta)$ is just this K_0-class for u_α and v_η.

Lemma 7.6. *Suppose* $(\xi + \eta - \eta\alpha)(x) \in (1/3, 2/3)$ *for all* $x \in X$. *Then we have*

$$[e(\alpha, \xi, \eta)] = [e(\alpha, \xi, 0)] - B_\alpha(\eta)$$

in $K_0(A)$.

Proof. Put

$$c = \inf\{|(\xi + \eta - \eta\alpha)(x)| : x \in X\} - \frac{1}{3}.$$

Then c is positive. Choose a sufficiently finer Kakutani-Rohlin partition

$$\mathcal{P} = \{X(v, k) : v \in V, 1 \leq k \leq h(v)\}$$

for (X, α). Let $R(\mathcal{P})$ be the roof set of $\mathcal{P}$. We may assume that $h(v)$ is large and

$$\sup\{|\eta(x) - \eta(y)| : x, y \in U\} < \frac{c}{2}$$

for every $U \in \widetilde{\mathcal{P}}$, where

$$\widetilde{\mathcal{P}} = \{X(v, k) : v \in V, 1 \leq k < h(v)\} \cup \{R(\mathcal{P})\}.$$

Take $x_U \in U$ for all $U \in \widetilde{\mathcal{P}}$ and define $\eta' \in C(X, \mathbb{Z})$ by $\eta'(x) = \eta(x_U)$ for $x \in U$. It is not hard to see that η and η' are homotopic in $H(\alpha, \xi)$. By Lemma 7.5, $[e(\alpha, \xi, \eta)] = [e(\alpha, \xi, \eta')]$ and $B_\alpha(\eta) = B_\alpha(\eta')$. Hence, by replacing η by η', we may assume that η is constant on each clopen set belonging to $\widetilde{\mathcal{P}}$. Furthermore, by adding a constant function, we may assume that $\eta(x) = 0$ for all $x \in R(\mathcal{P})$.

Note that $|(\eta - \eta\alpha)(x)|$ is less than $2/3 - 7/15 = 1/5$ for all $x \in X$. There is a unique lift $\tilde{\eta} \in C(X, \mathbb{R})$ of η such that $\tilde{\eta}(x) = 0$ for all $x \in R(\mathcal{P})$ and $|(\tilde{\eta} - \tilde{\eta}\alpha)(y)| < 1/5$ for all $y \in R(\mathcal{P})^c$. Moreover, for every $v \in V$, there exists an integer m_v such that

$$|m_v - \tilde{\eta}(x)| < \frac{1}{5}$$

for all $x \in X(v, 1)$. Therefore $B_\alpha(\eta)$ is equal to $\sum_{v \in V} -m_v[1_{X(v,1)}]$, and

$$|m_v| = |m_v - \tilde{\eta}(\alpha^{h(v)-1}(x))| \leq |m_v - \tilde{\eta}(x)| + \sum_{k=1}^{h(v)-1} |\tilde{\eta}(\alpha^{k-1}(x)) - \tilde{\eta}(\alpha^k(x))| < \frac{h(v)}{5},$$

where x is a point in $X(v, 1)$.

Fix $v_0 \in V$. Let $a : \{1, 2, \ldots, h(v_0)\} \to \mathbb{R}$ be a map such that $a(h(v_0)) = 0$, $|m_{v_0} - a(1)| < 11/30$ and $|a(k) - a(k+1)| < 11/30$ for all $k = 1, 2, \ldots, h(v_0) - 1$. Define a continuous map κ from X to $\mathbb{R}$ by

$$\kappa(x) = \begin{cases} a(k) & x \in X(v_0, k), k = 1, 2, \ldots, h(v_0) \\ \tilde{\eta}(x) & \text{otherwise.} \end{cases}$$

Put $\hat{\kappa}(x) = \kappa(x) + \mathbb{Z}$. Then $\hat{\kappa} \in C(X, \mathbb{T})$ belongs to $H(\alpha, \xi)$, because $7/15 - 11/30 = 1/10$. For $t \in [0, 1]$, put $\kappa_t = t\kappa + (1 - t)\tilde{\eta}$. Then it is not hard to see that $\hat{\kappa}_t$ gives a homotopy from η to $\hat{\kappa}$ in $H(\alpha, \xi)$.

At first, let us consider the case that m_{v_0} is positive. Define $a : \{1, 2, \ldots, h(v_0)\} \to \mathbb{R}$ by $a(1) = a(2) = m_{v_0} + \frac{1}{3}$, $a(3) = m_{v_0}$ and

$$a(k) = \frac{m_{v_0}(h(v_0) - k)}{h(v_0) - 3}$$

for every $k = 4, 5, \ldots, h(v_0)$. By using this map a, define $\kappa \in C(X, \mathbb{R})$ be as above. It follows that η and $\hat{\kappa}$ are homotopic in $H(\alpha, \xi)$. Let f_1 be the continuous function on $X \times \mathbb{T}$ defined by

$$f_1(x, t) = \begin{cases} 10(t - 1/3) & (x, t) \in X(v_0, 2) \times [1/3, 1/3 + 1/10] \\ 1 & (x, t) \in X(v_0, 2) \times [1/3 + 1/10, 2/3] \\ 1 - 10(t - 2/3) & (x, t) \in X(v_0, 2) \times [2/3, 2/3 + 1/10] \\ 0 & \text{otherwise.} \end{cases}$$

Let $U = X(v_0, 2) \times \mathbb{T}$ and put $f_2 = (1_U - f_1) \circ (\alpha \times R_\xi)^{-1}$. Define a continuous function g on $X \times \mathbb{T}$ by

$$g(x, t) = \begin{cases} -\sqrt{10(t - 1/3)(1 - 10(t - 1/3))} & (x, t) \in X(v_0, 2) \times \in [1/3, 1/3 + 1/10] \\ \sqrt{10(t - 2/3)(1 - 10(t - 2/3))} & (x, t) \in X(v_0, 2) \times \in [2/3, 2/3 + 1/10] \\ 0 & \text{otherwise.} \end{cases}$$

Then it can be verified that $e = gu^* + (f_1 + f_2) + ug$ is a projection and e is equivalent to 1_U: in fact, if $h \in C(X \times \mathbb{T})$ is a function with

$$h|X(v_0, 2) \times [1/3, 1/3 + 1/10] = -1, \quad h|X(v_0, 2) \times [2/3, 2/3 + 1/10] = 1$$

and $|h|^2 = 1$, then the partial isometry

$$w = h\sqrt{f_1} + \sqrt{1_U - f_1}u^*$$

satisfies $w^*w = e$ and $ww^* = 1_U$. Furthermore e is a subprojection of $e(\alpha, \xi, \hat{\kappa})$ and

$$e(\alpha, \xi, \hat{\kappa}) - e = e(\alpha, \xi, \eta')$$

for some $\eta' \in H(\alpha, \xi)$ with $B_\alpha(\eta') = B_\alpha(\eta) + [1_{X(v_0, 1)}]$. Hence

$$\begin{aligned} [e(\alpha, \xi, 0)] - [e(\alpha, \xi, \eta)] - B_\alpha(\eta) &= [e(\alpha, \xi, 0)] - [e(\alpha, \xi, \hat{\kappa})] - B_\alpha(\eta) \\ &= [e(\alpha, \xi, 0)] - [e(\alpha, \xi, \eta')] - [e] - B_\alpha(\eta) \\ &= [e(\alpha, \xi, 0)] - [e(\alpha, \xi, \eta')] - B_\alpha(\eta'). \end{aligned}$$

We can repeat the same argument with η' in place of η. By repeating this m_{v_0} times, we will obtain

$$[e(\alpha, \xi, 0)] - [e(\alpha, \xi, \eta)] - B_\alpha(\eta) = [e(\alpha, \xi, 0)] - [e(\alpha, \xi, \eta')] - B_\alpha(\eta')$$

with $B_\alpha(\eta') = \sum_{v \neq v_0} -m_v[1_{X(v, 1)}]$ and $\eta'(x) = 0$ for all $x \in X(v_0, 1) \cup \cdots \cup X(v_0, h(v_0))$.

When m_{v_0} is negative, there exists $\hat{\kappa}$ homotopic to η in $H(\alpha, \xi)$ such that $e(\alpha, \xi, \hat{\kappa}) + e = e(\alpha, \xi, \eta')$ for some $\eta' \in H(\alpha, \xi)$ with $B_\alpha(\eta') = B_\alpha(\eta) - [1_{X(v_0, 1)}]$. In a similar fashion to the preceding paragraph, the same conclusion follows.

By applying the same argument to all towers in V, we have

$$[e(\alpha, \xi, 0)] - [e(\alpha, \xi, \eta)] - B_\alpha(\eta) = 0.$$

$\square$

Lemma 7.7. *Let (X, α) be a Cantor minimal system. Suppose that $\xi_1, \xi_2 \in C(X, \mathbb{R})$ and $f \in C(X, \mathbb{Z})$ satisfy*

$$\mu(\xi_2) = \mu(\xi_1) + \mu(f)$$

for every α-invariant measure $\mu \in M_\alpha$ and

$$\frac{7}{15} < \xi_i(x) < \frac{8}{15}$$

for all $x \in X$ and $i = 1, 2$. Put $\hat{\xi}_i(x) = \xi_i(x) + \mathbb{Z}$ for $i = 1, 2$. Then, for any $\varepsilon > 0$, there exists $\eta \in C(X, \mathbb{T})$ such that

$$|(\hat{\xi}_1 - \hat{\xi}_2)(x) - (\eta - \eta\alpha)(x)| < \varepsilon$$

for all $x \in X$ and $B_\alpha(\eta) = [f]$.

Proof. We may assume $\varepsilon < 10^{-1}$. In the same way as in [GW, Lemma 2.4], we can find a Kakutani-Rohlin partition

$$\mathcal{P} = \{X(v, k) : v \in V, k = 1, 2, \ldots, h(v)\}$$

such that

$$\frac{1}{h(v)} \left| \sum_{k=1}^{h(v)} (\xi_1 - \xi_2 + f)(\alpha^{k-1}(x)) \right| < \varepsilon$$

for all $v \in V$ and $x \in X(v, 1)$. Define a real valued continuous function κ on $\alpha(R(\mathcal{P}))$ by

$$\kappa(x) = \sum_{k=1}^{h(v)} (\xi_1 - \xi_2 + f)(\alpha^{k-1}(x))$$

for $x \in X(v, 1)$. Let η be a real valued continuous function on X such that

$$\eta(\alpha^k(x)) = \sum_{i=1}^{k} (\xi_2 - \xi_1)(\alpha^{i-1}(x)) + \frac{k}{h(v)} \kappa(x)$$

for all $x \in X(v, 1)$ and $k = 0, 1, \ldots, h(v) - 1$. Define $\hat{\eta}(x) = \eta(x) + \mathbb{Z}$. It is straightforward to check

$$|(\hat{\xi}_1 - \hat{\xi}_2)(x) - (\hat{\eta} - \hat{\eta}\alpha)(x)| < \varepsilon$$

for all $x \in X$. For $x \notin R(\mathcal{P})$, we have

$$|(\eta - \eta\alpha)(x)| < \frac{1}{15} + \frac{1}{10},$$

and so the integer nearest to $(\eta - \eta\alpha)(x)$ is zero. If x belongs to the roof set $R(\mathcal{P})$, then

$$\left| (\eta - \eta\alpha)(x) - \sum_{k=1}^{h(v)} f(\alpha^{1-k}(x)) \right| < \frac{1}{15} + \frac{1}{10}.$$

Hence we can conclude that $B_\alpha(\eta)$ is equal to $[f]$. $\qquad\square$

Now we are ready to prove the main theorem of this section. Let (X, α) and (Y, β) be Cantor minimal systems and let $\xi : X \to \mathbb{T}$ and $\zeta : Y \to \mathbb{T}$ be continuous functions. Suppose that $\alpha \times R_\xi$ and $\beta \times R_\zeta$ are both minimal. We denote by A (resp. B) the crossed product C^*-algebra arising from $(X \times \mathbb{T}, \alpha \times R_\xi)$ (resp. $(Y \times \mathbb{T}, \beta \times R_\zeta)$).

Theorem 7.8. *The following are equivalent.*

(1) $(X \times \mathbb{T}, \alpha \times R_\xi)$ *is approximately K-conjugate to* $(Y \times \mathbb{T}, \beta \times R_\zeta)$.

(2) *There exists a unital order isomorphism ρ from $K_0(B)$ to $K_0(A)$ such that $\rho(K^0(Y, \beta)) = K^0(X, \alpha)$.*

Proof. (1)$\Rightarrow$(2). This is immediate from the definition of approximate K-conjugacy (Definition 7.3).

(2)$\Rightarrow$(1). By Lemma 6.1, without loss of generality, we may assume $\xi(x), \zeta(y) \in (7/15, 8/15)$ for all $x \in X$ and $y \in Y$. Let $e(\alpha, \xi, 0) \in A$ and $e(\beta, \zeta, 0) \in B$ be the projections described in the previous section. We follow the notation used there. Since ρ is an isomorphism, there are only two possibilities:

$$\rho([e(\beta, \zeta, 0)]) - [e(\alpha, \xi, 0)] \in K^0(X, \alpha)$$

or

$$\rho([e(\beta, \zeta, 0)]) + [e(\alpha, \xi, 0)] \in K^0(X, \alpha).$$

Suppose that the latter equality holds. The dynamical system $(X \times \mathbb{T}, \alpha \times R_\xi)$ is conjugate to $(X \times \mathbb{T}, \alpha \times R_{-\xi})$ via the mapping $(x, t) \mapsto (x, -t)$. This conjugacy induces an isomorphism between the corresponding C^*-algebras, which in turn yields an isomorphism between the K_0-groups. One can see that $[e(\alpha, \xi, 0)]$ is sent to $[1 - e(\alpha, -\xi, 0)]$ by this isomorphism. For this reason, by replacing $\alpha \times R_\xi$ by $\alpha \times R_{-\xi}$, we may always assume that there exists $h \in C(X, \mathbb{Z})$ such that

$$\rho([e(\beta, \zeta, 0)]) = [e(\alpha, \xi, 0)] + [h]_\alpha$$

in $K_0(A)$.

The restriction of ρ to $K^0(Y, \beta)$ is a unital order isomorphism onto $K^0(X, \alpha)$. By [LM, Theorem 5.4] or [M3, Theorem 3.4], there exist homeomorphisms $\sigma_n : X \to Y$ such that

$$\sigma_n \alpha \sigma_n^{-1} \to \beta$$

in $\mathrm{Homeo}(Y)$ and

$$[f \sigma_n]_\alpha = \rho([f]_\beta)$$

for all $f \in C(Y, \mathbb{Z})$ and $n \in \mathbb{N}$. We may assume

$$|\zeta \circ \beta^{-1} \sigma_n(x) - \zeta \circ \sigma_n \alpha^{-1}(x)| < \frac{1}{n}$$

for all $x \in X$ and $n \in \mathbb{N}$. As in Remark 4.6, we denote the state on the K_0-group arising from an invariant measure μ by S_μ. Then, for any $\mu \in M_\alpha$ and $n \in \mathbb{N}$, we have

$$\rho^*(S_\mu)([f]_\beta) = S_\mu(\rho([f]_\beta)) = \mu([f \sigma_n]_\alpha) = S_{\sigma_{n*}(\mu)}([f]_\beta)$$

for all $f \in C(Y, \mathbb{Z})$. Thus, $\rho^*(S_\mu) = S_{\sigma_{n*}(\mu)}$ on $K^0(Y, \beta)$, and so on $K_0(B)$ (see Remark 4.6). Let $\tilde{\xi} \in C(X, \mathbb{R})$ and $\tilde{\zeta} \in C(Y, \mathbb{R})$ be the lifts of ξ and ζ satisfying

$$\frac{7}{15} < \tilde{\xi}(x) < \frac{8}{15} \quad \text{and} \quad \frac{7}{15} < \tilde{\zeta}(y) < \frac{8}{15}$$

for all $x \in X$ and $y \in Y$. It follows that

$$
\begin{aligned}
\mu(\tilde{\zeta} \sigma_n) &= \sigma_{n*}(\mu)(\tilde{\zeta}) = S_{\sigma_{n*}(\mu)}([e(\beta, \zeta, 0)]) \\
&= \rho^*(S_\mu)([e(\beta, \zeta, 0)]) = S_\mu(\rho([e(\beta, \zeta, 0)])) \\
&= S_\mu([e(\alpha, \xi, 0)] + [h]_\alpha) \\
&= \mu(\tilde{\xi}) + \mu(h)
\end{aligned}
$$

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for every $\mu \in M_\alpha$ and $n \in \mathbb{N}$. Now Lemma 7.7 applies and yields $\eta_n \in C(X, \mathbb{T})$ such that

$$|(\xi - \zeta\sigma_n)(x) - (\eta_n - \eta_n\alpha)(x)| < \frac{1}{n}$$

for all $x \in X$ and $B_\alpha(\eta_n) = [h]$. Hence it is easy to verify that

$$(\sigma_n \times R_{\eta_n})(\alpha \times R_\xi)(\sigma_n \times R_{\eta_n})^{-1} \to (\beta \times R_\zeta)$$

in $\mathrm{Homeo}(Y \times \mathbb{T})$. Furthermore, since

$$|\zeta\beta^{-1}\sigma_n(x) - (\xi\alpha^{-1} + \eta_n - \eta_n\alpha^{-1})(x)| < \frac{2}{n}$$

for all $x \in X$, we get the estimate

$$\|f(\beta, \zeta, 0) \circ (\sigma_n \times R_{\eta_n}) - f(\alpha, \xi, -\eta_n)\| < \frac{20}{n}.$$

See the discussion before Proposition 6.4 for the definition of $f(\cdot, \cdot, \cdot)$. It is easy to see $g_0 \circ (\sigma_n \times R_{\eta_n}) = g_{-\eta_n}$. Therefore, the asymptotic morphism $\{\psi_n\} : B \to A$ associated with $\sigma_n \times R_{\eta_n}$ satisfies

$$\lim_{n \to \infty} \|\psi_n(e(\beta, \zeta, 0)) - e(\alpha, \xi, -\eta_n)\| = 0,$$

and Lemma 7.6 yields

$$[e(\alpha, \xi, -\eta_n)] = [e(\alpha, \xi, 0)] + B_\alpha(\eta_n)$$
$$= [e(\alpha, \xi, 0)] + [h]_\alpha = \rho([e(\beta, \zeta, 0)]).$$

For every clopen set $U \subset Y$, we know

$$\lim_{n \to \infty} \|\psi_n(1_U) - 1_U \circ \sigma_n\| = 0$$

and $[1_U \circ \sigma_n]_\alpha = \rho([1_U]_\beta)$. It follows that $\{\psi_n\}$ induces $\rho : K_0(B) \to K_0(A)$.

For every clopen set $U \subset Y$, we know

$$\lim_{n \to \infty} \|\psi_n(z1_U) - z1_U \circ (\sigma_n \times R_{\eta_n})\| = 0,$$

where z is a unitary defined by $z(y, t) = e^{2\pi\sqrt{-1}t}$. It is clear that

$$[1_{U^c} + z1_U \circ (\sigma_n \times R_{\eta_n})] = [1_{U^c} + z(1_U \circ \sigma_n)]$$

in $K_1(A)$. Since $\{\psi_n\}$ approximately carries the implementing unitary of $\beta \times R_\zeta$ to that of $\alpha \times R_\xi$, we can conclude that $\{\psi_n\}$ induces an isomorphism between $K_1(B)$ and $K_1(A)$.

Consequently $\{\psi_n\}$ induces an isomorphism between K-groups. Similarly, we can construct an asymptotic morphism from A to B which induces ρ^{-1} between their K-groups. $\qquad\square$

Theorem 7.9. *Suppose that $\alpha \times R_\xi$ and $\beta \times R_\zeta$ are minimal and rigid. Then the following are equivalent.*

(1) *$(X \times \mathbb{T}, \alpha \times R_\xi)$ is approximately K-conjugate to $(Y \times \mathbb{T}, \beta \times R_\zeta)$.*

(2) *There exists a unital order isomorphism ρ from $K_0(B)$ to $K_0(A)$ such that $\rho(K^0(Y, \beta)) = K^0(X, \alpha)$.*

(3) *$(X \times \mathbb{T}, \alpha \times R_\xi)$ is approximately flip K-conjugate to $(Y \times \mathbb{T}, \beta \times R_\zeta)$.*

(4) $\alpha \times R_\xi$ and $\beta \times R_\zeta$ are C^*-strongly approximately flip conjugate.

(5) There is $\theta \in KL(A, B)$ which gives an order and unit preserving isomorphism from $(K_0(A), K_0(A)_+, [1_A], K_1(A))$ onto $(K_0(B), K_0(B)_+, [1_B], K_1(B))$ and an isomorphism $\chi : C(X \times \mathbb{T}) \to C(Y \times \mathbb{T})$ such that

$$[j_{\alpha \times R_\xi}] \times \theta = [j_{\beta \times R_\zeta} \circ \chi]$$

in $KL(C(X \times \mathbb{T}), B)$.

Proof. We have seen (1)⇔(2) in Theorem 7.8.

(1)⇒(3) is obvious.

When both $\alpha \times R_\xi$ and $\beta \times R_\zeta$ are rigid, A and B have tracial rank zero by Theorem 5.6. Thus,(3)⇒(4) follows from [L6, Theorem 5.4].

(4)⇒(5) follows immediately from [L6, Theorem 3.9].

(5)⇒(2). Suppose that θ induces an order and unit preserving isomorphism $\Gamma(\theta)$ from $(K_0(A), K_0(A)_+, [1_A], K_1(A))$ to $(K_0(B), K_0(B)_+, [1_B], K_1(B))$. Suppose that there is $\chi : C(X \times \mathbb{T}) \to C(Y \times \mathbb{T})$ such that $[j_{\alpha \times R_\xi}] \times \theta = [j_{\beta \times R_\zeta} \circ \chi]$. This implies that $\Gamma(\theta)$ gives an isomorphism from $K^0(X, \alpha)$ onto $K^0(Y, \beta)$. So (2) holds. $\square$

8 Non-orientation preserving case

Let (X, α) be a Cantor minimal system and let $\varphi : X \to \text{Homeo}(\mathbb{T})$ be a continuous map. In this section, we would like to consider the case that $\alpha \times \varphi$ is not orientation preserving, that is, $[o(\varphi)]$ is not zero in $K^0(X, \alpha)/2K^0(X, \alpha)$. As was seen in the discussion before Lemma 2.5, the skew product extension $(X \times \mathbb{Z}_2, \alpha \times o(\varphi))$ is a Cantor minimal system. This system will play an important role when we study $\alpha \times \varphi$.

Define a continuous map $\widetilde{\varphi} : X \times \mathbb{Z}_2 \to \text{Homeo}(\mathbb{T})^+$ by

$$\widetilde{\varphi}_{(x,k)} = \lambda^{k+o(\varphi)(x)} \varphi_x \lambda^k,$$

where λ is given by $\lambda(t) = -t$ for $t \in \mathbb{T}$. Let π be the projection from $X \times \mathbb{Z}_2$ to the first coordinate.

Lemma 8.1. As a $\text{Homeo}(\mathbb{T})$-valued cocycle on the Cantor minimal system $(X \times \mathbb{Z}_2, \alpha \times o(\varphi))$, $\varphi\pi$ is cohomologous to $\widetilde{\varphi}$. In particular, $\varphi\pi$ is orientation preserving with respect to the minimal homeomorphism $\alpha \times o(\varphi)$.

Proof. Put $\omega_{(x.k)} = \lambda^k$. Then

$$\widetilde{\varphi}_{(x,k)} \circ \omega_{(x,k)} = \lambda^{k+o(\varphi)(x)} \circ \varphi_x = \omega_{(\alpha \times o(\varphi))(x,k)} \circ \varphi_{\pi(x,k)}$$

implies that they are cohomologous. $\square$

We remark that the following diagram of factor maps is commutative:

$$
\begin{array}{ccc}
(X \times \mathbb{T}, \alpha \times \varphi) & \xleftarrow{\ \pi \times \text{id}\ } & (X \times \mathbb{Z}_2 \times \mathbb{T}, \alpha \times o(\varphi) \times \varphi\pi) \\
\downarrow & & \downarrow \\
(X, \alpha) & \xleftarrow[\ \pi\ ]{} & (X \times \mathbb{Z}_2, \alpha \times o(\varphi)).
\end{array}
$$

From Lemma 6.5 and 6.6, there exists $\xi \in C(X, \mathbb{T})$ such that $r(R_{\xi(x)}\lambda^{o(\varphi)(x)}\varphi_x) \neq 0$ for all $x \in X$. By applying Lemma 6.2 to $-\xi$ and $o(\varphi)$, we obtain $\eta \in C(X, \mathbb{T})$ such that

$$|\eta(x) - (-1)^{o(\varphi)(x)}\eta\alpha(x) - \xi(x)| < \varepsilon,$$

where ε is sufficiently small so that it implies

$$
\begin{aligned}
0 \neq r(R_{\eta(x)-(-1)^{o(\varphi)(x)}\eta\alpha(x)}\lambda^{o(\varphi)(x)}\varphi_x) \\
= r(R^{-1}_{(-1)^{o(\varphi)(x)}\eta\alpha(x)}\lambda^{o(\varphi)(x)}\varphi_x R_{\eta(x)}) \\
= r(\lambda^{o(\varphi)(x)}R^{-1}_{\eta\alpha(x)}\varphi_x R_{\eta(x)})
\end{aligned}
$$

for all $x \in X$. Therefore, by perturbing φ by R_η, we may assume

$$r(\widetilde{\varphi}_{(x,k)}) = (-1)^k r(\lambda^{o(\varphi)(x)}\varphi_x) \neq 0$$

for all $(x, k) \in X \times \mathbb{Z}_2$.

Let A denote the crossed product C^*-algebra arising from $(X \times \mathbb{T}, \alpha \times \varphi)$. We write the implementing unitary by u. Define an automorphism $\theta \in \mathrm{Aut}(A)$ of order two by

$$\theta(f) = f$$

for all $f \in C(X \times \mathbb{T})$ and

$$\theta(u) = ug,$$

where $g \in C(X \times \mathbb{T})$ is given by $g(x, t) = (-1)^{o(\varphi)(x)}$. By Lemma 6.1, it can be seen that θ is approximately inner, and so it induces the identity on the K-group.

Proposition 8.2. *In the situation above, the crossed product C^*-algebra arising from the dynamical system $(X \times \mathbb{Z}_2 \times \mathbb{T}, \alpha \times o(\varphi) \times \varphi\pi)$ is isomorphic to $A \rtimes_\theta \mathbb{Z}_2$.*

Proof. We write the implementing unitary in A by u_A. Let us denote $C^*(X \times \mathbb{Z}_2 \times \mathbb{T}, \alpha \times o(\varphi) \times \varphi\pi)$ by B and the implementing unitary in B by u_B. Let v denote the unitary which implements θ.

We would like to define a homomorphism Φ from $A \rtimes \mathbb{Z}_2$ to B. The C^*-algebra A can be naturally embedded into B via the factor map $\pi \times \mathrm{id}$ from $X \times \mathbb{Z}_2 \times \mathbb{T}$ to $X \times \mathbb{T}$. Let $\Phi|A$ be this embedding. Define a continuous function $h \in C(X \times \mathbb{Z}_2 \times \mathbb{T})$ by

$$h(x, k, t) = (-1)^k.$$

Then,

$$h \circ (\alpha \times o(\varphi) \times \varphi\pi) = h(g \circ (\pi \times \mathrm{id})).$$

Put $\Phi(v) = h$. For $f \in C(X \times \mathbb{T})$, we have

$$h\Phi(f)h = h(f \circ (\pi \times \mathrm{id}))h = f \circ (\pi \times \mathrm{id}) = \Phi(\theta(f)).$$

Besides,

$$h\Phi(u_A)h = hu_Bh = u_B(g \circ (\pi \times \mathrm{id})) = \Phi(\theta(u_A)).$$

It follows that Φ is a well-defined homomorphism. It is not hard to see that Φ is an isomorphism.

$\square$

We freely use the identification of the two C^*-algebras established in the proposition above. By Lemma 8.1 and 2.4, we know that

$$K_0(A \rtimes \mathbb{Z}_2) \cong \mathbb{Z} \oplus K^0(X \times \mathbb{Z}_2, \alpha \times o(\varphi))$$

and

$$K_1(A \rtimes \mathbb{Z}_2) \cong \mathbb{Z} \oplus K^0(X \times \mathbb{Z}_2, \alpha \times o(\varphi)).$$

On the crossed product $C^*(X \times \mathbb{Z}_2 \times \mathbb{T}, \alpha \times o(\varphi) \times \varphi\pi)$, the dual action $\hat{\theta}$ is given by

$$\hat{\theta}(f)(x, k, t) = f(x, k+1, t)$$

for $f \in C(X \times \mathbb{Z}_2 \times \mathbb{T})$ and

$$\hat{\theta}(u) = u,$$

where u is the implementing unitary. Define a homeomorphism $\gamma \in \mathrm{Homeo}(X \times \mathbb{Z}_2)$ by

$$\gamma(x, k) = (x, k+1).$$

Then $\gamma \times \mathrm{id}$ commutes with $\alpha \times o(\varphi) \times \varphi\pi$ and $\hat{\theta}$ on $C(X \times \mathbb{Z}_2 \times \mathbb{T})$ is induced by $\gamma \times \mathrm{id}$. Let us consider the induced action $\hat{\theta}_*$ on the K-groups. Evidently, $\hat{\theta}_{*0}$ on $K^0(X \times \mathbb{Z}_2, \alpha \times o(\varphi))$ is given by

$$[f] \mapsto [f\gamma],$$

and $\hat{\theta}_{*1}$ on $K^0(X \times \mathbb{Z}_2, \alpha \times o(\varphi))$ is given by

$$[f] \mapsto [-f\gamma].$$

Of course, $\hat{\theta}_*([u]) = [u]$. Hence it remains to know the image of the generalized Rieffel projection. Let

$$e = e(\alpha \times o(\varphi), \widetilde{\varphi}, 0) = g_0 u^* + f(\alpha \times o(\varphi), \widetilde{\varphi}, 0) + u g_0$$

be the projection of $C^*(X \times \mathbb{Z}_2 \times \mathbb{T}, \alpha \times o(\varphi) \times \widetilde{\varphi})$ as in Proposition 6.8. This is well-defined, because $r(\widetilde{\varphi}_{(x,k)})$ is not zero. By the map

$$(x, k, t) \mapsto (x, k+1, \lambda(t)),$$

$f(\alpha \times o(\varphi), \widetilde{\varphi}, 0)$ is carried to a function supported on

$$\left\{ (x, k, t) : t \in [-\widetilde{\varphi}_{(\alpha \times o(\varphi))^{-1}(x, k+1)}(c), 0] \right\}$$

and g_0 is carried to a function supported on $X \times \mathbb{Z}_2 \times [-c, 0]$. Note that

$$
\begin{aligned}
&- \widetilde{\varphi}_{(\alpha \times o(\varphi))^{-1}(x, k+1)}(c) \\
&= -\lambda^{k+1} \varphi_{\alpha^{-1}(x)} \lambda^{k+1+o(\varphi)(\alpha^{-1}(x))}(c) \\
&= -\lambda \widetilde{\varphi}_{(\alpha \times o(\varphi))^{-1}(x, k)} \lambda(c) \\
&= \widetilde{\varphi}_{(\alpha \times o(\varphi))^{-1}(x, k)}(-c).
\end{aligned}
$$

Hence, under the identification of $C^*(X \times \mathbb{Z}_2 \times \mathbb{T}, \alpha \times o(\varphi) \times \widetilde{\varphi})$ with $C^*(X \times \mathbb{Z}_2 \times \mathbb{T}, \alpha \times o(\varphi) \times \varphi\pi)$, we have

$$
\begin{aligned}
\hat{\theta}(e) &= \hat{\theta}(e(\alpha \times o(\varphi), \widetilde{\varphi}, 0)) \\
&= g_{-c} u^* + 1 - f(\alpha \times o(\varphi), \widetilde{\varphi}, -c) + u g_{-c} \\
&= 1 - (-g_{-c} u^* + f(\alpha \times o(\varphi), \widetilde{\varphi}, -c) - u g_{-c}).
\end{aligned}
$$

By the remark following Proposition 6.8, this is homotopic to

$$1 - e(\alpha \times o(\varphi), \widetilde{\varphi}, -c),$$

and moreover it is homotopic to

$$1 - e(\alpha \times o(\varphi), \widetilde{\varphi}, 0)$$

from Proposition 6.8. Thus, we get

$$[\hat{\theta}(e)] = 1 - [e]$$

in $K_0(A \rtimes \mathbb{Z}_2)$. In particular, $\hat{\theta}$ is not approximately inner.

Next, we would like to consider the map between K-groups induced from the inclusion $\iota : A \hookrightarrow A \rtimes \mathbb{Z}_2$. On the K_0-group, that is clearly given by

$$\iota_{*0}([f]) = (0, [f \circ \pi]) \in \mathbb{Z} \oplus K^0(X \times \mathbb{Z}_2, \alpha \times o(\varphi)) \cong K_0(A \rtimes \mathbb{Z}_2)$$

for $[f] \in K^0(X, \alpha) \cong K_0(A)$. On the K_1-group, for $[f] \in \mathrm{Coker}(\mathrm{id} - \alpha_\varphi^*)$, we can see that

$$\iota_{*1}([f]) = (0, [\delta(f)]) \in \mathbb{Z} \oplus K^0(X \times \mathbb{Z}_2, \alpha \times o(\varphi)) \cong K_1(A \rtimes \mathbb{Z}_2),$$

where $\delta(f)(x, k) = (-1)^k f(x)$ for $(x, k) \in X \times \mathbb{Z}_2$. Under the identification

$$\mathrm{Coker}(\mathrm{id} - \alpha_\varphi^*) \cong K^0(X \times \mathbb{Z}_2, \alpha \times o(\varphi))/K^0(X, \alpha)$$

established in the discussion before Lemma 2.5, this map is also described by

$$\iota_{*1}([f] + K^0(X, \alpha)) = (0, [f - f\gamma])$$

for $f \in C(X \times \mathbb{Z}_2, \mathbb{Z})$.

We now consider minimality and rigidity of $\alpha \times \varphi$.

Lemma 8.3. *Let (X, α) be a Cantor minimal system and let $\varphi : X \to \mathrm{Homeo}(\mathbb{T})$ be a continuous map. Suppose that $\alpha \times \varphi$ is not orientation preserving. Then $\alpha \times \varphi$ is minimal if and only if $\alpha \times o(\varphi) \times \widetilde{\varphi}$ is minimal.*

Proof. We follow the notation used in the discussion above. By Lemma 8.1, we may replace $\alpha \times o(\varphi) \times \widetilde{\varphi}$ by $\alpha \times o(\varphi) \times \varphi\pi$.

Suppose that $\alpha \times o(\varphi) \times \varphi\pi$ is minimal. If $E \subset X \times \mathbb{T}$ is a closed $\alpha \times \varphi$-invariant subset, then $(\pi \times \mathrm{id})^{-1}(E)$ is a closed $\alpha \times o(\varphi) \times \varphi\pi$-invariant subset of $X \times \mathbb{Z}_2 \times \mathbb{T}$. It follows that $(\pi \times \mathrm{id})^{-1}(E)$ is empty or the whole $X \times \mathbb{Z}_2 \times \mathbb{T}$. Namely E is empty or the whole $X \times \mathbb{T}$.

Let us prove the converse. Assume that $\alpha \times \varphi$ is minimal. Let $E \subset X \times \mathbb{Z}_2 \times \mathbb{T}$ be a minimal subset of $\alpha \times o(\varphi) \times \varphi\pi$. Since $(\pi \times \mathrm{id})(E)$ is a closed $\alpha \times \varphi$-invariant subset, it must be equal to $X \times \mathbb{T}$. If $E = \gamma(E)$, then $E = X \times \mathbb{Z}_2 \times \mathbb{T}$ and we have nothing to do. Suppose that $E \cap \gamma(E)$ is empty and $E \cup \gamma(E) = X \times \mathbb{Z}_2 \times \mathbb{T}$. Thus E is a clopen subset. Hence there exists a continuous function $\chi : X \to \mathbb{Z}_2$ such that

$$E = \{(x, \chi(x), t) : x \in X, t \in \mathbb{T}\}.$$

It follows that $\chi + o(\varphi) = \chi\alpha$, which contradicts $[o(\varphi)] \neq 0$ in $K^0(X, \alpha)/2K^0(X, \alpha)$. $\square$

Lemma 8.4. *Let (X, α) be a Cantor minimal system and let $\varphi : X \to \mathrm{Homeo}(\mathbb{T})$ be a continuous map. Suppose that $\alpha \times \varphi$ is not orientation preserving.*

(1) *If $\alpha \times o(\varphi) \times \widetilde{\varphi}$ is rigid, then $\alpha \times \varphi$ is rigid.*

(2) *If φ takes its values in* $\mathrm{Isom}(\mathbb{T})$ *and* $\alpha \times \varphi$ *is rigid, then* $\alpha \times o(\varphi) \times \widetilde{\varphi}$ *is rigid.*

Proof. We follow the notation used in the discussion above. By Lemma 8.1, we may replace $\alpha \times o(\varphi) \times \widetilde{\varphi}$ by $\alpha \times o(\varphi) \times \varphi\pi$. Let F (resp. $\widetilde{F}$) denote the canonical factor map from $(X \times \mathbb{T}, \alpha \times \varphi)$ to (X, α) (resp. from $(X \times \mathbb{Z}_2 \times \mathbb{T}, \alpha \times o(\varphi) \times \varphi\pi)$ to $(X \times \mathbb{Z}_2, \alpha \times o(\varphi))$).

(1) Suppose that there exist two distinct ergodic measures ν_1 and ν_2 for $(X \times \mathbb{T}, \alpha \times \varphi)$ such that $F_*(\nu_1) = F_*(\nu_2)$. Let $\tilde{\nu}_i$ be an $\alpha \times o(\varphi) \times \varphi\pi$-invariant measure such that $(\pi \times \mathrm{id})_*(\tilde{\nu}_i) = \nu_i$. Of course, $\tilde{\nu}_1 \neq \tilde{\nu}_2$, because of

$$(\pi \times \mathrm{id})_*(\tilde{\nu}_1) = \nu_1 \neq \nu_2 = (\pi \times \mathrm{id})_*(\tilde{\nu}_2).$$

By replacing $\tilde{\nu}_i$ by

$$\frac{1}{2}(\tilde{\nu}_i + (\gamma \times \mathrm{id})_*(\tilde{\nu}_i)),$$

we may assume that $\tilde{\nu}_i$ is $\gamma \times \mathrm{id}$-invariant. It follows that $\widetilde{F}_*(\tilde{\nu}_i)$ is invariant under γ. Together with

$$\pi_* \widetilde{F}_*(\tilde{\nu}_1) = F_*(\pi \times \mathrm{id})_*(\tilde{\nu}_1) = F_*(\nu_1) = F_*(\nu_2) = F_*(\pi \times \mathrm{id})_*(\tilde{\nu}_2) = \pi_* \widetilde{F}_*(\tilde{\nu}_2),$$

we have $\widetilde{F}_*(\tilde{\nu}_1) = \widetilde{F}_*(\tilde{\nu}_2)$. Therefore $\alpha \times o(\varphi) \times \varphi\pi$ is not rigid.

(2) Assume that $\alpha \times o(\varphi) \times \varphi\pi$ is not rigid. There exists an ergodic measure μ for $(X \times \mathbb{Z}_2, \alpha \times o(\varphi))$ such that $\widetilde{F}_*^{-1}(\mu)$ is not a singleton. By assumption, $\widetilde{\varphi}$ takes its values in the rotation group. It follows from Lemma 4.4 and its proof that there exist uncountably many ergodic measures for $(X \times \mathbb{Z}_2 \times \mathbb{T}, \alpha \times o(\varphi) \times \varphi\pi)$ in $\widetilde{F}_*^{-1}(\mu)$. In particular, we can find two distinct ergodic measures $\nu_1, \nu_2 \in \widetilde{F}_*^{-1}(\mu)$ such that $(\gamma \times \mathrm{id})_*(\nu_1) \neq \nu_2$. Hence, it is easily verified that

$$(\pi \times \mathrm{id})_*(\nu_1) \neq (\pi \times \mathrm{id})_*(\nu_2)$$

in $M_{\alpha \times \varphi}$. But, we have

$$F_*(\pi \times \mathrm{id})_*(\nu_1) = \pi_* \widetilde{F}_*(\nu_1) = \pi_*(\mu) = \pi_* \widetilde{F}_*(\nu_2) = F_*(\pi \times \mathrm{id})_*(\nu_2),$$

and so $\alpha \times \varphi$ is not rigid. $\qquad\square$

Now we consider cocycles with values in $\mathrm{Isom}(\mathbb{T})$. Let (X, α) be a Cantor minimal system and let $\varphi : X \to \mathrm{Isom}(\mathbb{T})$ be a continuous map. There exists $\xi \in C(X, \mathbb{T})$ such that $\varphi_x = \lambda^{o(\varphi)(x)} R_{\xi(x)}$ for all $x \in X$. Suppose that $\alpha \times \varphi$ is not orientation preserving and not minimal. By Lemma 8.3, $\alpha \times o(\varphi) \times \widetilde{\varphi}$ is not minimal, where $\widetilde{\varphi}$ is given by

$$\widetilde{\varphi}_{(x,k)} = \lambda^{k+o(\varphi)(x)} \varphi_x \lambda^k = \lambda^k R_{\xi(x)} \lambda^k = \begin{cases} R_{\xi(x)} & k = 0 \\ R_{-\xi(x)} & k = 1. \end{cases}$$

It is convenient to introduce $\delta : C(X, \mathbb{T}) \to C(X \times \mathbb{Z}_2, \mathbb{T})$ defined by $\delta(\eta)(x, k) = (-1)^k \eta(x)$. It follows from Lemma 4.2 that there exist $n \in \mathbb{N}$ and $\zeta \in C(X \times \mathbb{Z}_2, \mathbb{Z})$ such that

$$n\delta(\xi) = \zeta - \zeta \circ (\alpha \times o(\varphi))^{-1}.$$

We also have

$$-n\delta(\xi) = n\delta(\xi) \circ \gamma = (\zeta - \zeta \circ (\alpha \times o(\varphi))^{-1}) \circ \gamma = \zeta\gamma - \zeta\gamma \circ (\alpha \times o(\varphi))^{-1}.$$

Since $\alpha \times o(\varphi)$ is minimal on $X \times \mathbb{Z}_2$, $\zeta + \zeta\gamma$ must equal a constant function. We can adjust ζ by a constant function so that $\zeta + \zeta\gamma$ is equal to zero. Thus there exists $\eta \in C(X, \mathbb{T})$ such that

$$n\delta(\xi) = \delta(\eta) - \delta(\eta) \circ (\alpha \times o(\varphi))^{-1}.$$

Combining this with $\delta \circ \alpha_\varphi^* = (\alpha \times o(\varphi))^* \circ \delta$, we obtain

$$n\xi = \eta - \alpha_\varphi^*(\eta).$$

Lemma 8.5. *Let (X, α) be a Cantor minimal system and let $\varphi_x = \lambda^{o(\varphi)(x)} R_{\xi(x)}$ be a cocycle with values in $\mathrm{Isom}(\mathbb{T})$. If $\alpha \times \varphi$ is not orientation preserving and not minimal, then there exist $n \in \mathbb{N}$ and $\eta \in C(X, \mathbb{T})$ such that*

$$n\xi = \eta - \alpha_\varphi^*(\eta).$$

Moreover, every minimal subset of $\alpha \times \varphi$ is given by

$$E_s = \{(x, t) : nt = \alpha_\varphi^*(\eta)(x) + s \ \ or \ \ nt = \alpha_\varphi^*(\eta)(x) - s\}$$

for some $s \in \mathbb{T}$.

Proof. The first part follows the discussion above. Let us consider the latter part. By Lemma 4.2, every minimal set of $\alpha \times o(\varphi) \times \widetilde{\varphi}$ is given by

$$\{(x, k, t) : nt = (-1)^{k+o(\varphi)(\alpha^{-1}(x))} \eta \alpha^{-1}(x) + s\}.$$

This closed set is carried to

$$E_s = \{(x, t) : nt = \alpha_\varphi^*(\eta)(x) + s \ \ or \ \ nt = \alpha_\varphi^*(\eta)(x) - s\}$$

by the factor map from $X \times \mathbb{Z}_2 \times \mathbb{T}$ to $X \times \mathbb{T}$. Therefore E_s is a minimal set of $\alpha \times \varphi$. $\square$

9 Examples

Example 9.1. Let $\theta \in \mathbb{T}$ be an irrational number and let $\xi : \mathbb{T} \to \mathbb{T}$ be a continuous map. A homeomorphism

$$\gamma : (s, t) \mapsto (s + \theta, t + \xi(s))$$

on $\mathbb{T}^2$ is called a Furstenberg transformation. In [OP], the crossed product C^*-algebra arising from $(\mathbb{T}^2, \gamma)$ is studied. We would like to replace the irrational rotation with a Cantor minimal system and construct an almost one to one extension of $(\mathbb{T}^2, \gamma)$ as follows. Let φ be a Denjoy homeomorphism on $\mathbb{T}$ with $r(\varphi) = \theta$ as in Remark 3.2. The homeomorphism φ has the unique invariant nontrivial closed subset X. Let α be the restriction of φ to X. Then (X, α) is a Cantor minimal system and there exists an almost one to one factor map π from (X, α) to the irrational rotation $(\mathbb{T}, R_\theta)$ (see [PSS] for details). Both (X, α) and $(\mathbb{T}, R_\theta)$ are uniquely ergodic. It is easy to see that $\pi \times \mathrm{id} : X \times \mathbb{T} \to \mathbb{T}^2$ satisfies

$$(\pi \times \mathrm{id}) \circ (\alpha \times R_{\xi\pi}) = \gamma \circ (\pi \times \mathrm{id}),$$

that is, $\pi \times \mathrm{id}$ is a factor map. One can check that if γ is minimal, then $\alpha \times R_{\xi\pi}$ is also minimal. The factor map π induces a Borel isomorphism between X and S^1. Hence $\alpha \times R_{\xi\pi}$ is rigid if and only if γ is uniquely ergodic. There are some known criterions for unique ergodicity of γ. For example, it was proved in [F, Theorem 2.1] that if ξ is a Lipschitz function and its degree is not zero then γ is uniquely ergodic. On the other hand it is known that there exist $\theta \in \mathbb{T}$ and $\xi : \mathbb{T} \to \mathbb{T}$ such that γ is minimal but not uniquely ergodic (see [F, p.585] for instance). In this case $\alpha \times R_{\xi\pi}$ is minimal but not rigid.

Now let us consider the example of Putnam which was presented by N. C. Phillips in [Ph3].

Example 9.2. For any $\theta \in \mathbb{R} \setminus \mathbb{Q}$ let g_θ be a minimal homeomorphism of a Cantor set $X_\theta \subset \mathbb{T}$ obtained from a Denjoy homeomorphism $g_\theta^0 : \mathbb{T} \to \mathbb{T}$ as in [Ph3]. Choose g_θ^0 to have rotation number θ and such that the unique minimal set $X_\theta \subset \mathbb{T}$ has the property that the image of $\mathbb{T} \setminus X_\theta$ under the semiconjugation to R_θ is a single orbit of R_θ.

Now let $\theta_1, \theta_2 \in \mathbb{R} \setminus \mathbb{Q}$ be irrational numbers such that $1, \theta_1, \theta_2$ are $\mathbb{Q}$-linearly independent. Consider two systems $(X_{\theta_1} \times \mathbb{T}, g_{\theta_1} \times R_{\theta_2})$ and $(X_{\theta_2} \times \mathbb{T}, g_{\theta_2} \times R_{\theta_1})$. Then both are minimal. Let $A = C^*(X_{\theta_1} \times \mathbb{T}, g_{\theta_1} \times R_{\theta_2})$ and $B = C^*(X_{\theta_2} \times \mathbb{T}, g_{\theta_2} \times R_{\theta_1})$. It follows from [Ph3, Proposition 1.12] that

$$K_1(A) \cong K_1(B) = \mathbb{Z}^3$$

and

$$(K_0(A), K_0(A)_+, [1_A]) \cong (\mathbb{Z} + \theta_1\mathbb{Z} + \theta_2\mathbb{Z}, (\mathbb{Z} + \theta_1\mathbb{Z} + \theta_2\mathbb{Z})_+, 1) \cong (K_0(B), K_0(B)_+, [1_B]),$$

where $\mathbb{Z} + \theta_1\mathbb{Z} + \theta_2\mathbb{Z} \subset \mathbb{R}$. In particular, both systems are rigid. Moreover A and B have tracial rank zero and they are isomorphic by the classification theorem in [L5]. However, there is no order isomorphism between $\mathbb{Z} + \theta_1\mathbb{Z}$ and $\mathbb{Z} + \theta_2\mathbb{Z}$. It follows from Theorem 7.9 that they are not approximately K-conjugate. On the other hand, by [M3, Corollary 4.10], they are weakly approximately conjugate.

When we choose another continuous function $\xi : X_{\theta_1} \to \mathbb{T}$, two systems $g_{\theta_1} \times R_{\theta_2}$ and $g_{\theta_1} \times R_\xi$ may not be conjugate. But, if the integral value of ξ is equal to θ_2, then we can conclude that they are approximately K-conjugate by Theorem 7.8.

Example 9.3. We would like to construct a non-orientation preserving minimal homeomorphism on $X \times \mathbb{T}$ concretely. Let (X, α) be an odometer system of type 3^∞. It is well-known that $K^0(X, \alpha)$ is isomorphic to $\mathbb{Z}[1/3]$. We regard X as a projective limit of $\mathbb{Z}_{3^n}$ and denote the canonical projection from X to $\mathbb{Z}_{3^n}$ by π_n. Notice that $\pi_n(\alpha(x)) = \pi_n(x) + 1$ for all $x \in X$, where the addition is understood modulo 3^n. Let x_0 be the point of X such that $\pi_n(x_0) = 0$ for all $n \in \mathbb{N}$. We will construct a continuous map $\varphi : X \to \mathrm{Isom}(\mathbb{T})$ of the form $\varphi_x = \lambda R_{\xi(x)}$ so that $\alpha \times \varphi$ is a minimal homeomorphism on $X \times \mathbb{T}$. Note that $o(\varphi)(x) = 1$ and $[o(\varphi)] \neq 0$ in $K^0(X, \alpha)/2K^0(X, \alpha) \cong \mathbb{Z}_2$. By Lemma 8.5, if the closure of

$$\{(\alpha \times \varphi)^m(x_0, 0) : m \in \mathbb{N}\}$$

contains $\{x_0\} \times \mathbb{T}$, then we can deduce the minimality of $\alpha \times \varphi$. Let $\{t_n\}_{n \in \mathbb{N}}$ be a dense sequence of $\mathbb{T}$. Since $\alpha^{3^n}(x_0) \to x_0$ as $n \to \infty$, it suffices to construct φ so that $(\alpha \times \varphi)^{3^n}(x_0, 0) = (\alpha^{3^n}(x_0), -t_n)$.

Let $s_n \in (-2^{-1}, 2^{-1}]$ be the real number satisfying $s_n + \mathbb{Z} = t_n - t_{n-1}$, where we put $t_0 = 0$. We define a map $\xi_n : X \to \mathbb{T}$ by

$$\xi_n(x) = \begin{cases} 0 & \pi_n(x) = 0, 1, \ldots, 3^{n-1} - 1 \\ \frac{(-1)^k}{2 \cdot 3^{n-1}} s_n + \mathbb{Z} & \text{otherwise.} \end{cases}$$

For all $n, m \in \mathbb{N}$, it is not hard to see that

$$\sum_{k=0}^{3^n - 1} (-1)^k \xi_m(\alpha^k(x_0)) = \begin{cases} 0 & n < m \\ s_m + \mathbb{Z} & n \geq m. \end{cases}$$

Since $|\xi_n(x)| < 3^{-n}$ for every $x \in X$,

$$\xi(x) = \sum_{n=1}^{\infty} \xi_n(x)$$

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exists and is continuous on X. Put $\varphi_x = \lambda R_{\xi(x)}$ for all $x \in X$. Then

$$\sum_{k=0}^{3^n-1} (-1)^k \xi(\alpha^k(x_0)) = \sum_{i=1}^{n} s_i + \mathbb{Z} = t_n$$

implies

$$(\alpha \times \varphi)^{3^n}(x_0, 0) = (\alpha^{3^n}(x_0), \lambda(t_n)) = (\alpha^{3^n}(x_0), -t_n)$$

for all $n \in \mathbb{N}$. It follows that $\alpha \times \varphi$ is minimal.

Example 9.4. We would like to construct a cocycle with values in $\mathrm{Homeo}^+(\mathbb{T})$ which is not cohomologous to a cocycle with values in the rotation group. It is useful to introduce a complete metric $d(\cdot, \cdot)$ of $\mathrm{Homeo}^+(\mathbb{T})$ defined by

$$d(\varphi, \psi) = \max_{t \in \mathbb{T}} \left\{ |\varphi(t) - \psi(t)|, |\varphi^{-1}(t) - \psi^{-1}(t)| \right\}.$$

In the argument below, we use the following facts.

Fact (a). For any $s, t \in \mathbb{T}$ and $\varepsilon > 0$, there exists $\rho \in \mathrm{Homeo}^+(\mathbb{T})$ such that $|\rho(s) - s| < \varepsilon$, $|\rho(t) - s| < \varepsilon$ and ρ is conjugate to an irrational rotation.

Fact (b). $\mathrm{Homeo}^+(\mathbb{T})$ is arcwise connected.

Let us construct two sequences of natural numbers m_n and l_n, and a sequence of maps $\varphi_n : \mathbb{Z}_{m_n} \to \mathrm{Homeo}^+(\mathbb{T})$ inductively so that the following conditions are satisfied. It is convenient to view φ_n as a periodic map from $\mathbb{Z}$.

(1) m_{n-1} divides m_n and l_n is not greater than m_n/m_{n-1}.

(2) $\psi_n = \varphi_n(m_n - 1) \ldots \varphi_n(1) \varphi_n(0)$ is conjugate to an irrational rotation.

(3) Both $|\psi_n(0)|$ and $|\psi_n(1/2)|$ are less than $1/n$.

(4) For every $t \in \mathbb{T}$, $\{\psi_{n-1}^k(t) : k = 1, 2, \ldots, l_n\}$ is $1/n$-dense in $\mathbb{T}$.

(5) For every $k = 0, 1, \ldots, l_n m_{n-1} - 1$, $\varphi_n(k) = \varphi_{n-1}(k)$.

(6) For every $k \in \mathbb{Z}_{m_n}$, $d(\varphi_n(k), \varphi_{n-1}(k))$ is less than 2^{-n}.

If these conditions are achieved, then we can finish the proof as follows. Let (X, α) be the odometer system of type $\{m_n\}_n$. Namely, X is the projective limit of $\mathbb{Z}_{m_n}$ and there exists a natural projection $\pi_n : X \to \mathbb{Z}_{m_n}$ such that $\pi_n(\alpha(x)) = \pi_n(x) + 1$, where the addition is understood modulo m_n. Let x_0 be the point of X such that $\pi_n(x_0) = 0$ for all $n \in \mathbb{N}$. From (6),

$$\varphi_x = \lim_{n \to \infty} \varphi_n(\pi_n(x)) \in \mathrm{Homeo}^+(\mathbb{T})$$

exists for all $x \in X$ and φ is a continuous map from X to $\mathrm{Homeo}^+(\mathbb{T})$. By (4) and (5), for all $n \in \mathbb{N}$ and $t \in \mathbb{T}$, we can see that

$$\{\varphi_{\alpha^{k-1}(x_0)} \ldots \varphi_{\alpha(x_0)} \varphi_{x_0}(t) : k = m_{n-1}, 2m_{n-1}, \ldots, l_n m_{n-1}\}$$

is $1/n$-dense in $\mathbb{T}$. Hence, for every $t \in \mathbb{T}$, the closure of

$$\{(\alpha \times \varphi)^k(x_0, t) : k \in \mathbb{N}\}$$

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in $X \times \mathbb{T}$ contains $\{x_0\} \times \mathbb{T}$. It follows that $\alpha \times \varphi$ is minimal. Let us check that φ is never cohomologous to a cocycle with values in the rotation group. Suppose that φ is cohomologous to a cocycle with values in the rotation group. Then we would have a homeomorphism $\mathrm{id} \times \gamma$ on $X \times \mathbb{T}$ such that

$$(\mathrm{id} \times \gamma)(\alpha \times \varphi) = (\alpha \times \varphi)(\mathrm{id} \times \gamma)$$

and $(\mathrm{id} \times \gamma)(x_0, 0) = (x_0, 1/2)$. By (3), we can verify that

$$(\alpha \times \varphi)^{m_n}(x_0, 0) \to (x_0, 0)$$

and

$$(\alpha \times \varphi)^{m_n}(x_0, 1/2) \to (x_0, 0)$$

in $X \times \mathbb{T}$ as $n \to \infty$. Consequently we obtain

$$
\begin{aligned}
(\mathrm{id} \times \gamma)(x_0, 0) &= \lim_{n \to \infty} (\mathrm{id} \times \gamma)(\alpha \times \varphi)^{m_n}(x_0, 0) \\
&= \lim_{n \to \infty} (\alpha \times \varphi)^{m_n}(\mathrm{id} \times \gamma)(x_0, 0) \\
&= \lim_{n \to \infty} (\alpha \times \varphi)^{m_n}(x_0, 1/2) = (x_0, 0),
\end{aligned}
$$

which is a contradiction.

Let us construct m_n, l_n and φ_n. Put $m_1 = l_1 = 1$ and let φ_1 be an irrational rotation. Suppose that m_{n-1}, k_{n-1} and φ_{n-1} have been fixed. Put

$$\psi_{n-1} = \varphi_{n-1}(m_{n-1} - 1) \ldots \varphi_{n-1}(1)\varphi_{n-1}(0).$$

By (2), there exists $\omega \in \mathrm{Homeo}^+(\mathbb{T})$ such that $\omega \psi_{n-1} \omega^{-1} = R_\theta$, where $r(\psi_{n-1}) = \theta$ is an irrational number. Since ψ_{n-1} is minimal, we can find a natural number l_n so that the condition (4) holds. Applying Fact (a) to $\omega(0)$ and $\omega(1/2)$, we obtain $\rho \in \mathrm{Homeo}^+(\mathbb{T})$ such that both $|\omega^{-1}\rho\omega(0)|$ and $|\omega^{-1}\rho\omega(1/2)|$ are less than $1/n$. Furthermore ρ is conjugate to an irrational rotation. Define $\tilde{\omega} : \mathbb{Z}_{m_{n-1}} \to \mathrm{Homeo}^+(\mathbb{T})$ by

$$\tilde{\omega}(k) = \omega \varphi_{n-1}(0)^{-1} \varphi_{n-1}(1)^{-1} \ldots \varphi_{n-1}(k-1)^{-1}$$

for all $k = 0, 1, \ldots, m_{n-1} - 1$. Evidently we have

$$\tilde{\omega}(k+1)\varphi_{n-1}(k)\tilde{\omega}(k)^{-1} = \begin{cases} \mathrm{id} & k \neq m_{n-1} - 1 \\ R_\theta & k = m_{n-1} - 1 \end{cases}$$

for all $k \in \mathbb{Z}_{m_{n-1}}$. Choose $\varepsilon > 0$ so that $d(\varphi, \mathrm{id}) < \varepsilon$ implies

$$d(\varphi_{n-1}(k)\tilde{\omega}(k)^{-1}\varphi\tilde{\omega}(k), \varphi_{n-1}(k)) < \frac{1}{2^n}$$

for all $k \in \mathbb{Z}_{m_{n-1}}$. By Fact (b), we can find a natural number N greater than l_n and a sequence of homeomorphisms

$$\mathrm{id} = \rho_0, \rho_1, \ldots, \rho_{m_{n-1}(N-l_n)} = \rho R_\theta^{-N}$$

such that

$$d(\rho_{i+1}\rho_i^{-1}, \mathrm{id}) < \varepsilon$$

for all $i = 0, 1, \ldots, m_{n-1}(N - l_n) - 1$. Note that this is easily done because $d(\cdot, \cdot)$ is invariant under rotations. Put $m_n = m_{n-1}N$. We define a map $\tilde{\rho} : \mathbb{Z}_{m_n} \to \mathrm{Homeo}^+(\mathbb{T})$ by

$$\tilde{\rho}(k) = \begin{cases} \mathrm{id} & k = 0, 1, \ldots, l_n m_{n-1} - 1 \\ R_\theta^{-j}\rho_{k'+1}\rho_{k'}^{-1}R_\theta^j & \text{otherwise,} \end{cases}$$

where $k' = k - l_n m_{n-1}$ and j is a natural number satisfying $N - k/m_{n-1} \le j < N + 1 - k/m_{n-1}$. Notice that we still have

$$d(\tilde{\rho}(k), \mathrm{id}) < \varepsilon$$

for all $k \in \mathbb{Z}_{m_n}$. Define $\varphi_n : \mathbb{Z}_{m_n} \to \mathrm{Homeo}^+(\mathbb{T})$ by

$$\varphi_n(k) = \varphi_{n-1}(k)\tilde{\omega}(k)^{-1}\tilde{\rho}(k)\tilde{\omega}(k).$$

The condition (5) is already built in this definition. The condition (6) is immediate from the choice of ε. Since one can check that

$$
\begin{aligned}
\psi_n &= \varphi_n(m_n - 1)\ldots\varphi_n(1)\varphi_n(0) \\
&= \tilde{\omega}(m_n)^{-1}\left(R_\theta \tilde{\rho}(m_n - 1)\ldots\tilde{\rho}(m_n - m_{n-1})R_\theta\ldots\tilde{\rho}(l_n m_{n-1})\right)R_\theta^{l_n}\tilde{\omega}(0) \\
&= \omega^{-1}\rho R_\theta^{-N}R_\theta^N\omega \\
&= \omega^{-1}\rho\omega,
\end{aligned}
$$

the conditions (2) and (3) follow immediately.

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